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MATHEMATICAL METHODS UNITS 3 & 4

TRIAL EXAMINATION 2

(ANALYSIS TASK)

2005

Reading Time: 15 minutes
Writing time: 90 minutes

Instructions to students

This exam consists of 4 questions.
All questions should be answered.
There is a total of 55 marks available.
The marks allocated to each of the four questions are indicated throughout.
Students may bring up to two A4 pages of pre-written notes into the exam.
Formula sheets and a table of the Normal distribution - cdf can be found on pages 15-17 of this exam.

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Question 1

"Powball" is a lottery open to the public and drawn each Thursday night. A government supervisor must be present to supervise each Powball draw. Four supervisors, George, Danielle, Bhawani and Tim have supervised the draws on the proportion of occasions indicated in the table below where $p > 0$.

Supervisor	Proportion of draws supervised
George	$p - q$
Danielle	p^2
Bhawani	$p^2 + p$
Tim	$4q$

a. **i.** Show that $2p(p+1) = 1 - 3q$.

ii. Given that $q = \frac{1}{8}$ and $p > 0$, calculate the value of p and find the probability that on a particular night when a Powball draw took place, George was the supervisor.

1 + 2 = 3 marks

The Powball lottery involves 25 identically sized balls that are labelled 1 through 25 and are mixed up in a large cylinder that rotates. A ball is randomly selected and sent down a chute then a second and a third ball are randomly selected and follow the first ball down the chute. These three balls provide the winning numbers for the Powball lottery.

b. **i.** What is the probability that the first ball down the chute is an odd numbered ball?

ii. Find the exact probability that of the three balls chosen, exactly two are odd.

iii. If fifteen instead of three balls were chosen in the same manner, find the probability that exactly seven are odd numbered. Express your answer correct to four decimal places.

1 + 2 + 2 = 5 marks

One Thursday night the Powball machine malfunctioned. Before the second ball entered the chute, the first ball drawn fell out of the chute back in with all the other balls and became mixed in with them. This happened three times in total before the machine was switched off.

c. **i.** Find the probability that the ball numbered sixteen appeared exactly once during this time. Express your answer correct to 4 decimal places.

ii. Find the probability that the number sixteen appears at least once during this time. Express your answer correct to 4 decimal places.

iii. The machine is switched back on but the same problem continues. There is a single ball chosen randomly on n occasions before the machine is switched off again.
The probability that no balls numbered with a single digit are chosen just prior to the machine being switched off is 0.044.
Find the value of n.

1 + 2 + 2 = 5 marks

d. The weight of the balls that are supplied by the manufacturers to the Powball organizers is normally distributed with a mean of 15g and a standard deviation of 1.5g.
Because of the way in which the balls are mixed in the machine, the organizers of Powball only use balls that weigh between 14g and 16g.
What percentage of balls, correct to 1 decimal place, supplied by the manufacturers are actually used by the Powball organizers?

2 marks
Total 15 marks

Question 2

In a window display, a horizontal shelf is connected to a vertical pole. The shelf moves vertically up and down so that the distance d in centimetres from the floor below, at time t seconds is given by

$$d(t) = 120 + 25\sin\left(\frac{\pi t}{300}\right), \qquad t \geq 0.$$

The time $t = 0$ corresponds to 9am when the mechanism that moves the shelf is turned on.

a. What is the minimum distance of the shelf from the floor?

1 mark

b. What vertical distance does the shelf move through from its maximum height to its minimum height?

1 mark

c. What distance above the floor is the shelf at time $t = 75$ seconds?
Express your answer as an exact value.

1 mark

d. During the first ten minutes of movement, when is the shelf exactly one metre above the floor? Express your answers in seconds correct to 2 decimal places.

2 marks

e. At what rate is the distance of the shelf above the floor changing at $t = 50$ seconds? Express your answer correct to 2 decimal places.

2 marks

f. During the first ten minutes, when was the rate of change of the height of the shelf above the floor negative?

2 marks

g. A second shelf in the same window display also moves vertically up and down so that its height, h_1, in centimetres above the floor at time *t* seconds is given by

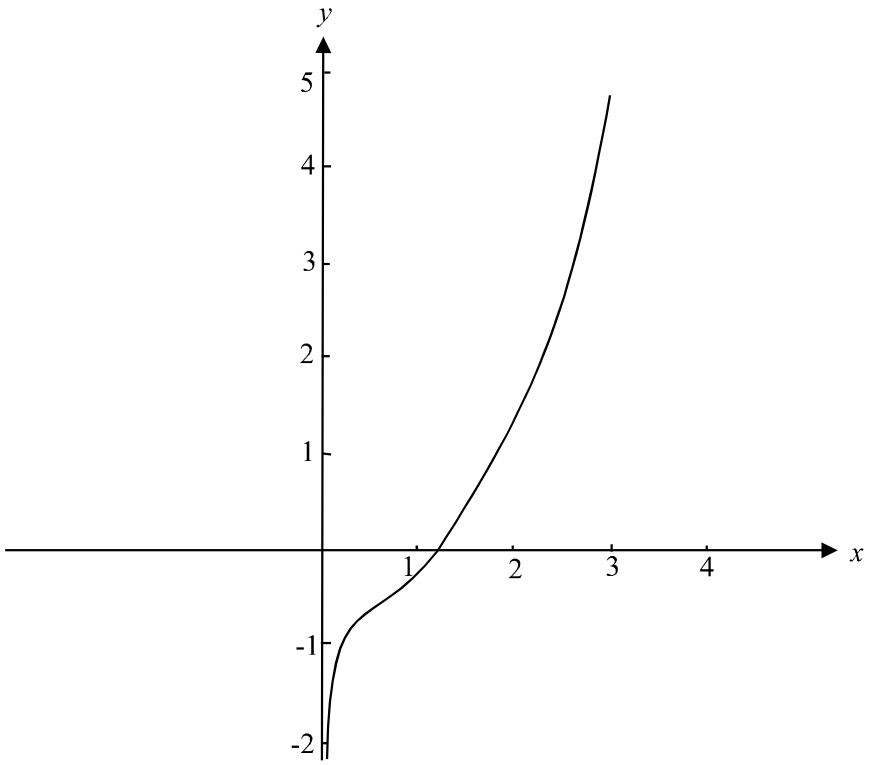
$$h_1(t) = 120 + 50\cos\left(\frac{\pi t}{300}\right)$$

The time $t = 0$ for this shelf also corresponds to 9am. By using algebra, find when the two shelves are at the same height above the floor during their first ten minutes of movement. Express your answer/s correct to 2 decimal places.

3 marks
Total 12 marks

Question 3

The graph of $y = f(x)$ where $f(x) = \log_e(2x) - 2x + x^2$, $x > 0$ is shown below.



a. Find the *x*-intercept of the graph. Express your answer correct to two decimal places.

1 mark

b. Explain why the function *f* is not defined for $x \leq 0$.

1 mark

c. Explain why the inverse function, $f^{-1}(x)$, exists.

1 mark

d. Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as the graph of $y = f(x)$ is given.

1 mark

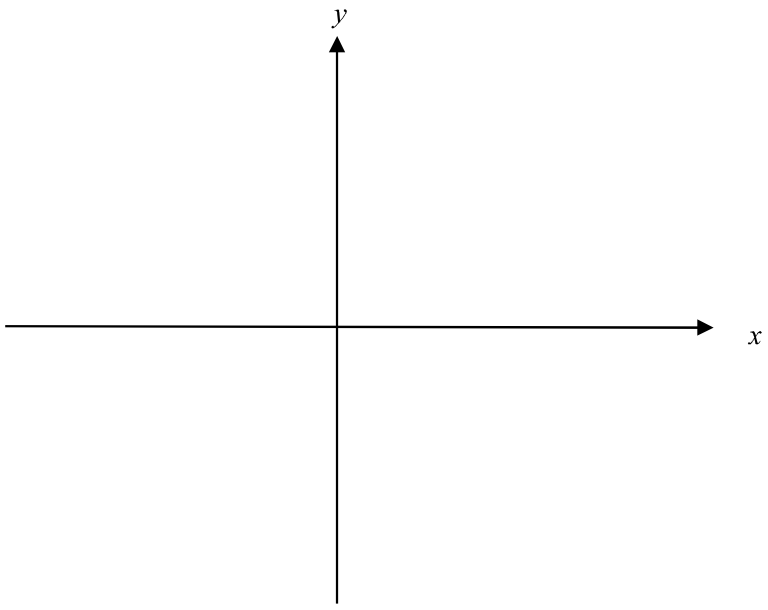
e. What is the *y*-intercept of the graph $y = f^{-1}(x)$? Express your answer correct to two decimal places.

1 mark

f. Use calculus to explain why the graph of $y = f(x)$ has no stationary points.

2 marks

g. Use calculus and a sketch graph to show that the rate of change of $f(x)$ with respect to x is always positive.



2 marks

h. The area enclosed by the *x*-axis, the *y*-axis and the normal to the curve $y = f(x)$ at the point where $x = 3$ is given by

$$\frac{1}{2} \times (a \log_e(6) + b) \times (\log_e(6) + c).$$

Find the values of *a*, *b* and *c*.

5 marks
Total 14 marks

Question 4

Scientists conducting a research project into mice plagues, set up colonies of mice. Each colony lives under different environmental conditions and each colony starts with the same number of mice. The number of mice in each colony at time *t* years after the colonies are set up is given by *N* where

$$N(t) = 50 \times e^{(t^2 - qt + 1)}$$

where $t \geq 0$ and $q \geq 0$.

a. Find the initial number of mice in each colony. Express your answer to the nearest whole number.

1 mark

b. By referring to the function *N(t)*, explain why the population of a colony can never equal zero.

1 mark

c. If $q = 0$ for a particular colony, show; using calculus, that the population of that colony can only increase.

2 marks

d. Show that if the population of a particular colony has a minimum population of 50 then $q = 2$.

4 marks

e. Sketch the graphs of $N = 50 \times e^{t^2 - 2t + 1}$ and $N = 50$ on the set of axes below for the domain $t \in [0, 2 \cdot 5]$

2 marks

f. By considering the intersection of the graphs with equation $N = 50 \times e^{t^2 - qt + 1}$ and $N = 50$, find the values of q for which the population of a mice colony would drop below 50.

4 marks
Total 14 marks

Table 1 Normal distribution – cdf

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

Mathematical Methods Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \qquad \int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x} \qquad \int \frac{1}{x} dx = \log_e(x) + c, \text{ for } x > 0$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax) \qquad \int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax) \qquad \int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Statistics and Probability

$\Pr(A) = 1 - \Pr(A')$ $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A / B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$ variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Discrete distributions

	$\Pr(X = x)$	mean	variance
general	$p(x)$	$\mu = \Sigma x\, p(x)$	$\sigma^2 = \Sigma (x-\mu)^2 p(x)$ $= \Sigma x^2 p(x) - \mu^2$
binomial	${}^nC_x p^x (1-p)^{n-x}$	np	$np(1-p)$
hypergeometric	$\dfrac{{}^DC_x \, {}^{N-D}C_{n-x}}{{}^NC_n}$	$n\dfrac{D}{N}$	$n\dfrac{D}{N}\left(1-\dfrac{D}{N}\right)\left(\dfrac{N-n}{N-1}\right)$

Continuous distributions

normal	If X is distributed $\text{N}(\mu, \sigma^2)$ and $Z = \dfrac{X-\mu}{\sigma}$ then Z is distributed $\text{N}(0,1)$.