THE HEFFERNAN GROUP

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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2005

1. C	**8. D**	**15. C**	**22. C**
2. E	**9. A**	**16. A**	**23. A**
3. A	**10. D**	**17. E**	**24. D**
4. E	**11. B**	**18. A**	**25. D**
5. B	**12. E**	**19. B**	**26. E**
6. E	**13. D**	**20. D**	**27. D**
7. C	**14. B**	**21. D**	

Part I – Multiple-choice solutions

Question 1

A discrete random variable is one that can be counted not measured. Depth, height, distance and weight are all measurements and so only the number of goals is discrete; that is, it could be 0, or 1 or 2 or so on.
The answer is C.

Question 2

This is a binomial distribution since there is office mail or there is no office mail. There is also a sample of 5.
The mean of the binomial distribution is np where $n = 5$ and $p = 0 \cdot 1$ (the probability of there being no office mail on a given day)

So $\quad np = 5 \times 0 \cdot 1$

$\qquad\quad = 0 \cdot 5$

The variance $= np(1 - p)$

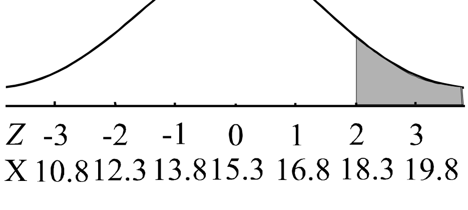
$\qquad\qquad = 0 \cdot 5(1 - 0 \cdot 1)$

$\qquad\qquad = 0 \cdot 45$

The answer is E.

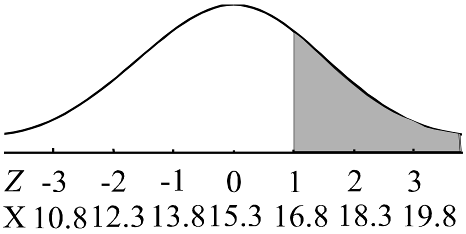
Question 3

The area shaded represents $\Pr(X > 18 \cdot 3)$.
$\Pr(X > 18 \cdot 3) = \Pr(z < -2)$ because of the symmetrical nature of the normal or the bell-shaped curve.

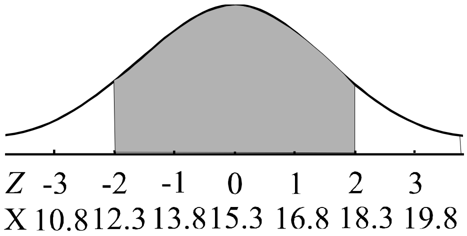


The areas representing the other options are shown

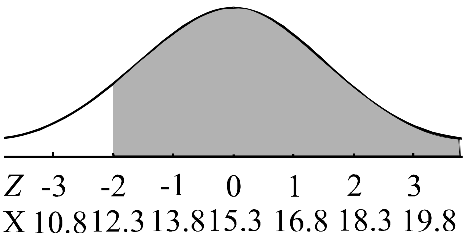
Option B



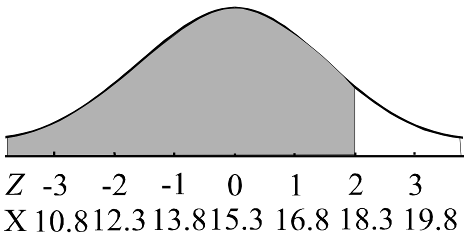
Option C



Option D



Option E



The answer is A.

Question 4

This is a hypergeometric distribution since there is no replacement (of donuts).

Method 1 – using the formula

$$\Pr(X = x) = \frac{{}^{D}C_{x}\ {}^{N-D}C_{n-x}}{{}^{N}c_{n}}$$

Now, $N = 12$, Let $D = 5$ (This is the number with pink icing. You could let $D = 7$ and you would get the same result.)

$x = 1,\ n = 2$

$$\Pr(X = 1) = \frac{{}^{5}C_{1}\ {}^{7}C_{1}}{{}^{12}C_{2}}$$

$$= 5 \times 7 \div \frac{12 \times 11}{2 \times 1}$$

$$= \frac{35}{66}$$

The answer is E.

Method 2

1 pink donut can be chosen from 5 pink donuts in ${}^5C_1$ ways.

Similarly 1 chocolate donut can be chosen from 7 chocolate donuts in ${}^7C_1$ ways.

2 donuts can be chosen from 12 donuts in ${}^{12}C_2$ ways.

The probability of choosing a pink and a chocolate donut is

$$\frac{{}^5C_1 \times {}^7C_1}{{}^{12}C_2}$$

$$= 5 \times 7 \div \frac{12 \times 11}{2 \times 1}$$

$$= \frac{35}{66}$$

The answer is E.

Question 5

We have a Binomial distribution with $p = 0 \cdot 25, q = 0 \cdot 75,\ n = 20$ and $x = 1$.

$$\begin{aligned} \Pr(X = x) &= {}^nC_x p^x q^{n-x} \\ \Pr(X = 1) &= {}^{20}C_1 (0 \cdot 25)^1 (0 \cdot 75)^{19} \\ &= 20 \times 0 \cdot 25 \times (0 \cdot 75)^{19} \\ &= 5 \times (0 \cdot 75)^{19} \end{aligned}$$

The answer is B.

Question 6

The shape is that of a sine function that has been translated 1 unit up.

The period is 12.

$$\begin{aligned} \text{Now period} &= \frac{2\pi}{n} \\ 12 &= \frac{2\pi}{n} \\ 12n &= 2\pi \\ n &= \frac{2\pi}{12} \\ &= \frac{\pi}{6} \end{aligned}$$

So the general equation

$y = \sin(nx) + 1$

becomes

$y = \sin\left(\frac{\pi x}{6}\right) + 1$

Note that the amplitude is 1.

The answer is E.

Question 7

$\sin\left(\frac{x}{3}\right)=\frac{1}{\sqrt{2}} \qquad 0 \le x \le 3\pi \quad \text{so} \quad 0 \le \frac{x}{3} \le \pi$

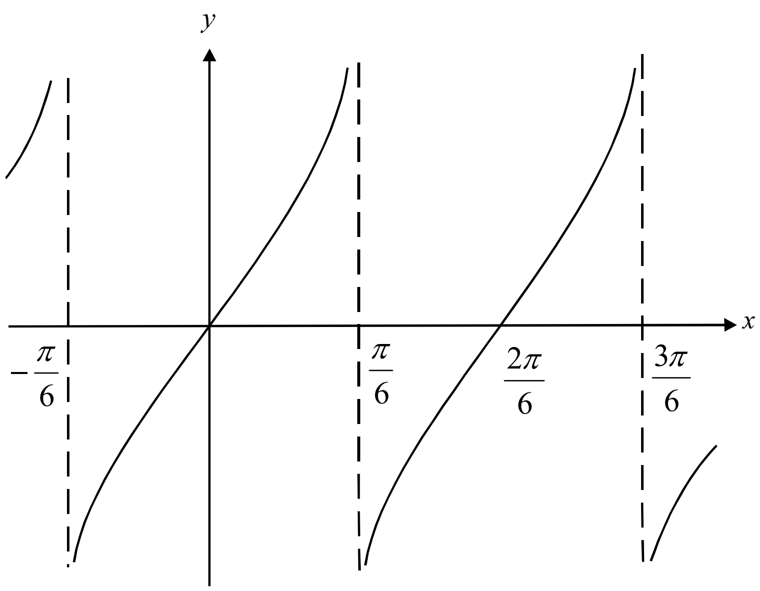
$$\frac{x}{3}=\frac{\pi}{4}, \frac{3\pi}{4}$$
$$x=\frac{3\pi}{4}, \frac{9\pi}{4}$$

S	A
T	C

So the sum is $\frac{12\pi}{4}=3\pi$

The answer is C.

Question 8



If two adjacent asymptotes occur at $x=\frac{\pi}{6}$ and $x=\frac{\pi}{2}$ or $x=\frac{3\pi}{6}$, then the period of the graph is $\frac{3\pi}{6}-\frac{\pi}{6}=\frac{2\pi}{6}=\frac{\pi}{3}$.

Now the period for a tan graph is $\frac{\pi}{a}$

So $\quad \frac{\pi}{3}=\frac{\pi}{a}$

$\quad a=3$

The answer is D.

Question 9

Use Pascal's triangle.

$$\begin{array}{c}1\\1\quad 1\\1\quad 2\quad 1\\1\quad 3\quad 3\quad 1\\1\quad 4\quad 6\quad 4\quad 1\\1\quad 5\quad 10\quad 10\quad 5\quad 1\\1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1\end{array}$$

The x^2 term is $15\times(ax)^2\times 2^4 = 6\,000x^2$

So $\quad 240a^2 = 6\,000$

$a^2 = 25$

$a = \pm 5$

The only correct answer offered is option A.
The answer is A.

Question 10

$$x^4 + 3x^3 - 2x^2 - 2x$$
$$= x(x^3 + 3x^2 - 2x - 2)$$

Let $P(x) = x^3 + 3x^2 - 2x - 2$

$P(1) = 1 + 3 - 2 - 2 = 0$

So $\quad x - 1 \quad$ is a factor.

$$\begin{aligned}P(x) &= x^3 + 3x^2 - 2x - 2\\ &= x^2(x-1) + \_(x-1) + \_(x-1)\\ &= x^2(x-1) + 4x(x-1) + \_(x-1)\\ &= x^2(x-1) + 4x(x-1) + 2(x-1)\\ &= (x-1)(x^2 + 4x + 2)\\ &= (x-1)\{(x^2 + 4x + 4) - 2\}\\ &= (x-1)((x+2)^2 - 2)\\ &= (x-1)(x + 2 - \sqrt{2})(x + 2 + \sqrt{2})\end{aligned}$$

The linear factors of $x^4 + 3x^3 - 2x^2 - 2x$ are $x, x-1, x+2-\sqrt{2}, x+2+\sqrt{2}$.
The answer is D.

Question 11

Let $y = 10^{\log_{10}(x-3)}$

Now $\log_{10} y = \log_{10}(x-3)$

So $y = x - 3$

So $10^{\log_{10}(x-3)} = x - 3$

The answer is B.

Question 12

$$\log_a(x) - \log_a(2x^2) = 3\log_a(x) + 1$$
$$\log_a\left(\frac{x}{2x^2}\right) = \log_a(x^3) + 1$$
$$\log_a\left(\frac{1}{2x}\right) - \log_a(x^3) = 1$$
$$\log_a\left(\frac{1}{2x} \div \frac{x^3}{1}\right) = 1$$
$$\log_a\left(\frac{1}{2x^4}\right) = 1$$
$$a^1 = \frac{1}{2x^4}$$
$$2ax^4 = 1$$
$$x^4 = \frac{1}{2a}$$
$$x = \sqrt[4]{\frac{1}{2a}}$$

The answer is E.

Question 13

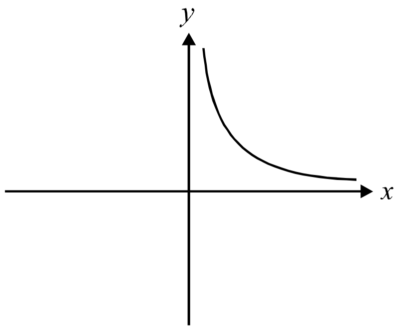
Graph the two functions for $x \geq 0$. Find the point of intersection. It is $(1\cdot 24, -3\cdot 46)$ correct to 2 decimal places in each case.
So the solution to the equation $-e^x = x^2 - 5$ is closest to $1\cdot 24$.
The answer is D.

Question 14

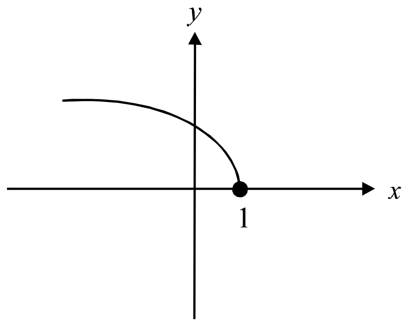
If the graph touches at $x = -1$, there must be a repeated factor i.e. $(x+1)^2$. This rules out options C and D. The polynomial is of degree 3 so this rules out options A and E.
The only feasible option is B.
The answer is B.

Question 15

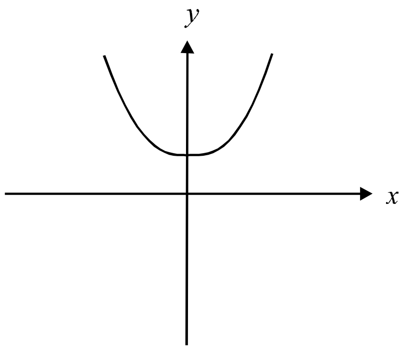
A.



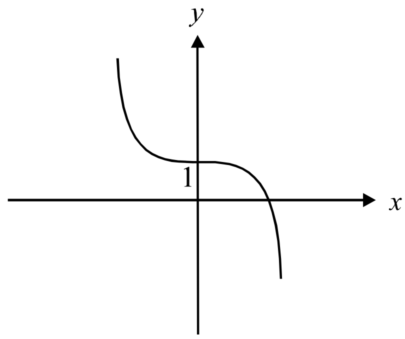
B.



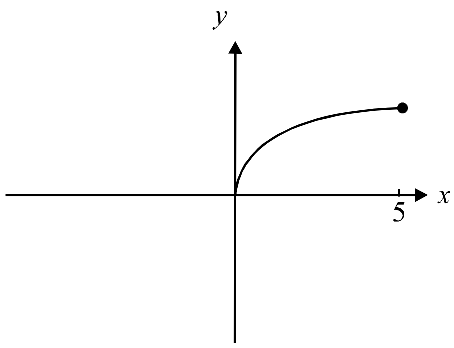
C.



D.



E.



All the graphs are shown. Option C shows a graph that is many:1 not 1:1.
The answer is C.

Question 16

We have $f:(-\infty,2)\to R,\ f(x)=\dfrac{3}{x-2}+1$

Let $y=\dfrac{3}{x-2}+1$

Swap x and y.

$x=\dfrac{3}{y-2}+1$

Rearrange

$$x-1=\frac{3}{y-2}$$

$$(x-1)(y-2)=3$$

$$y-2=\frac{3}{x-1}$$

$$y=\frac{3}{x-1}+2$$

Now, $d_f=r_{f^{-1}}$ and $r_f=d_{f^{-1}}$.

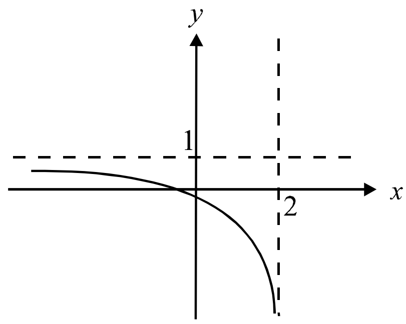
A quick sketch (note that $d_f=(-\infty,2)$) shows us that $r_f=(-\infty,1)$

so, $d_{f^{-1}}=(-\infty,1)$

So we have,

$f^{-1}:(-\infty,1)\to R,\ f^{-1}(x)=\dfrac{3}{x-1}+2$

The answer is A.



Question 17

The dilation by a factor of 2 from the x-axis means that the rule $y=\dfrac{1}{x^2}$ becomes $y=\dfrac{2}{x^2}$.

The subsequent dilation by a factor of 3 from the y-axis means that the rule becomes

$$y=\frac{2}{\left(\frac{x}{3}\right)^2}$$

$$=\frac{2}{x^2}\times 9$$

$$=\frac{18}{x^2}$$

So $g(x)=\dfrac{18}{x^2}$.

The answer is E.

Question 18

As $x \to -\infty,\ f'(x) \to 0$
As $x \to \infty,\ f'(x) \to \infty$
Also the gradient of the graph of $y = f(x)$ is always positive so options B, C and E are out.
The answer is A.

Question 19

$$y = 3\tan\left(\frac{x}{2}\right)$$
$$\frac{dy}{dx} = 3 \times \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$
$$= \frac{3}{2}\sec^2\left(\frac{x}{2}\right)$$
The answer is B.

Question 20

$y = \log_e(\cos(3x))$
Let $y = \log_e(u)$ where $u = \cos(3x)$
$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = -3\sin(3x)$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{Chain rule}$$
$$= \frac{1}{u} \cdot -3\sin(3x)$$
$$= \frac{1}{\cos(3x)} \cdot -3\sin(3x)$$
$$= -3\tan(3x)$$
The answer is D.

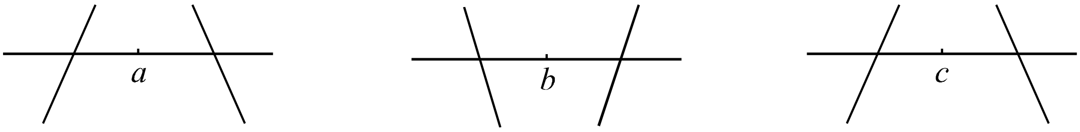
Question 21
$$y = \frac{e^{2x}}{\sin(x)}$$
$$\frac{dy}{dx} = \frac{\sin(x) \cdot 2e^{2x} - e^{2x}\cos(x)}{\sin^2(x)}$$
$$= \frac{e^{2x}(2\sin(x) - \cos(x))}{\sin^2(x)}$$
When $x = \frac{\pi}{2}$
$$\frac{dy}{dx} = \frac{e^{\pi}(2-0)}{1^2}$$
$$= 2e^{\pi}$$
The answer is D.

Question 22

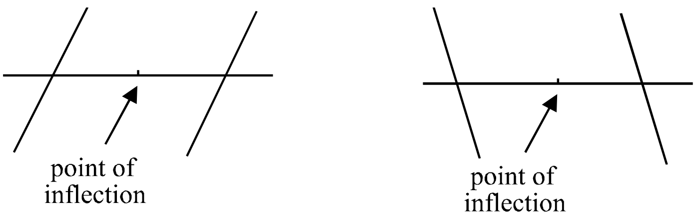
$f(x+h) \approx f(x) + hf'(x)$
and $h = 0 \cdot 01$
So $f(x+h) \approx f(3) + 0 \cdot 01 f'(3)$
The answer is C.

Question 23

The function $f'(x)$ is the derivative of the function $f(x)$. That is, the function $f'(x)$ is the gradient function of $f(x)$. At the points on the x-axis where $x = a, b$ and c, the gradient is zero. So at the corresponding points on the graph of $y = f(x)$ there will be three stationary points. None of these can be points of inflection since the sign of the gradient changes from positive to negative on either side of the points where $x = a$ and $x = c$ and from negative to positive on either side of the point where $x = b$. This indicates a local maximum at $x = a$ and at $x = c$ and a local minimum at $x = b$ as indicated in the diagrams below.



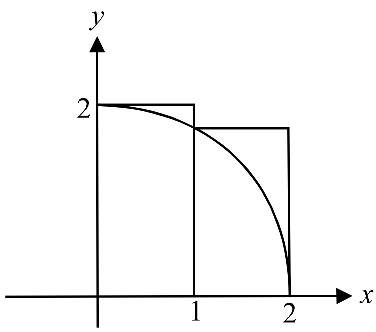
For a point of inflection with zero gradient to occur, the gradient to either side of the point of inflection would be positive or it would be negative as indicated in the diagrams below.



The answer is A.

Question 24

The left rectangle approximation is given by
$1 \times f(0) + 1 \times f(1)$
$= 1 \times \sqrt{4 - 0^2} + 1 \times \sqrt{4 - 1^2}$
$= \sqrt{4} + \sqrt{3}$
$= 2 + \sqrt{3}$
The answer is D.



Question 25

$$\int_{-1}^{2}(1-3f(x))\,dx=\int_{-1}^{2}1\,dx-3\int_{-1}^{2}f(x)\,dx$$
$$=[x]_{-1}^{2}-3\int_{-1}^{2}f(x)\,dx$$
$$=(2--1)-3\int_{-1}^{2}f(x)\,dx$$
$$=3-3\int_{-1}^{2}f(x)\,dx$$

The answer is D.

Question 26

$$\int_{0}^{n}(3-2x)\,dx=-4$$

So $\left[3x-x^2\right]_0^n=-4$

$(3n-n^2)-(0)=-4$

$3n-n^2+4=0$

$n^2-3n-4=0$

$(n-4)(n+1)=0$

$n=4$ or $n=-1$ but $n>0$

So $n=4$

The answer is E.

Question 27

The graphs of $y=\cos(x)$ and $y=\dfrac{1}{2}$ intersect when

$\cos(x)=\dfrac{1}{2}$

$x=\dfrac{\pi}{3},\dfrac{5\pi}{3}$

S	A
T	C

So the area shaded

$$=\int_{0}^{\frac{\pi}{3}}\left(\cos(x)-\frac{1}{2}\right)dx+\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\left(\frac{1}{2}-\cos(x)\right)dx+\int_{\frac{5\pi}{3}}^{2\pi}\left(\cos(x)-\frac{1}{2}\right)dx$$

$$=\int_{0}^{\frac{\pi}{3}}\left(\cos(x)-\frac{1}{2}\right)dx-\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}\left(\cos(x)-\frac{1}{2}\right)dx+\int_{\frac{5\pi}{3}}^{2\pi}\left(\cos(x)-\frac{1}{2}\right)dx$$

The answer is D.

PART II

Question 1

Let $P(x)=x^4+ax^3+9x^2-ax-5$

Now $P(-2)=16-8a+36+2a-5$ **(1 mark)**

So, $16-8a+36+2a-5=5$

$-6a+47=5$

$-6a=-42$

$a=7$

(1 mark)

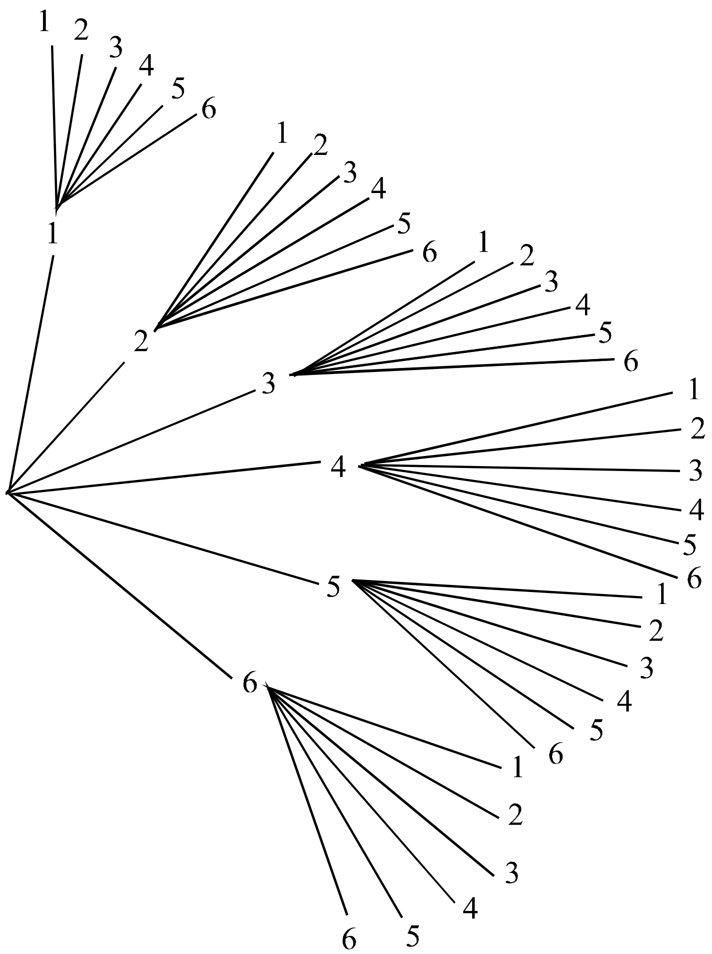
Question 2

a.

Number of sixes (x)	Probability $\Pr(X=x)$
0	$\frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$
1	$\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}+\frac{1}{6}=\frac{10}{36}$
2	$\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$

(2 marks)

Note that the probabilities must add to give 1.
Note also that a tree diagram can be used to obtain the probabilities.



Alternatively, in working out the probability of 1 six occurring, remember that a 6 can occur on the first throw or on the second throw.
That is,

$$\begin{aligned}&\Pr(1\text{ six occurs})\\&=\Pr(6,\text{ not }6)+\Pr(\text{not }6,\,6)\\&=\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\\&=\frac{10}{36}\end{aligned}$$

b. Method 1

$$\begin{aligned}\mu&=\sum x\,p(x)\\&=0\times\frac{25}{36}+1\times\frac{10}{36}+2\times\frac{1}{36}\\&=\frac{1}{3}\end{aligned}$$

(1 mark)

Method 2

Since we have a binomial distribution (throwing a six or not throwing a six and 2 tosses of the die) we can use the formula

$$\begin{aligned}\mu&=np\\&=2\times\frac{1}{6}\\&=\frac{1}{3}\end{aligned}$$

(1 mark)

c. Method 1

$$\begin{aligned}\sigma^2&=\sum x^2p(x)-\mu^2\\&=0^2\times\frac{25}{36}+1^2\times\frac{10}{36}+2^2\times\frac{1}{36}-\left(\frac{1}{3}\right)^2\\&=\frac{14}{36}-\frac{1}{9}\\&=\frac{14}{36}-\frac{4}{36}\\&=\frac{10}{36}\\&=\frac{5}{18}\end{aligned}$$

(1 mark)

Method 2

Since we have a binomial distribution we can use the formula

$$\begin{aligned}\sigma^2&=npq\\&=2\times\frac{1}{6}\times\frac{5}{6}\\&=\frac{5}{18}\end{aligned}$$

(1 mark)

Question 3

a. The period of the function *f* is 6. Note that the period is one complete cycle (i.e. trough to trough or peak to peak) whereas the graph $y = f(x)$ extends beyond one complete cycle.

(1 mark)

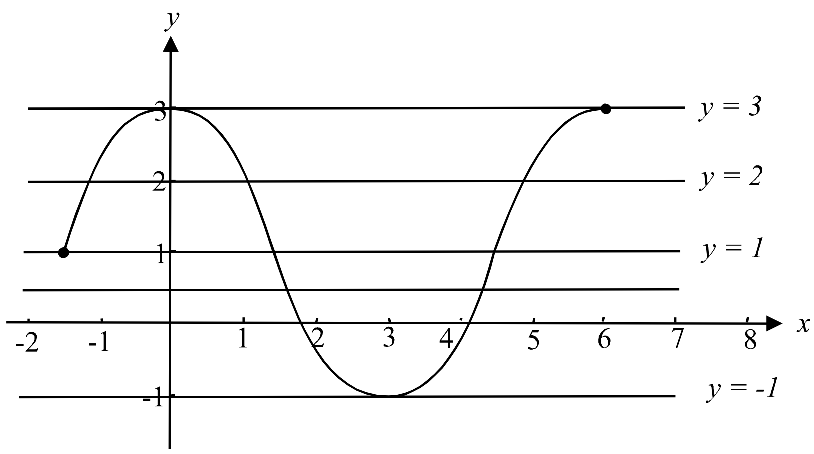
b. The amplitude is 2.

(1 mark)

c. The domain is $x \in [-1 \cdot 5, 6]$.

(1 mark)

d. The equation $f(x) = b$ is represented graphically as the intersection of the graph $y = f(x)$ and $y = b$. The graph of $y = b$ is a horizontal line. Use your ruler to impersonate this horizontal line and move it up and down the graph of $y = f(x)$.



Note that for $y = -1$ there is one point of intersection. Immediately above there are two points of intersection. Keep moving your horizontal ruler up and there remain two points of intersection until you get $y = 1$ where there are 3 points of intersection. Keep moving your horizontal ruler up until you get to $y = 3$ where there are 2 points of intersection again. So, there are two solutions to the equation $f(x) = b$ for $b \in (-1, 1) \cup [3, 3]$. Alternatively for $-1 < b < 1$ and $b = 3$.

(1 mark) $-1 < b < 1$
(1 mark) for $b = 3$

Question 4

a. Average rate of change of *h* with respect to *t* over $[0,5]$

$$= \frac{h(5)-h(0)}{5-0}$$ **(1 mark)**

$$= 26$$ **(1 mark)**

b. Instantaneous rate of change $= h'(t)$

$$h(t) = 2(t-1)^3 + 1$$

$$h'(t) = 2 \times 3(t-1)^2 \times 1 \quad \text{Chain rule}$$

$$= 6(t-1)^2$$ **(1 mark)**

When $h'(t) = 54$

$$6(t-1)^2 = 54$$

$$(t-1)^2 = 9$$

$$(t-1)^2 - 9 = 0$$

$$(t-1-3)(t-1+3) = 0$$

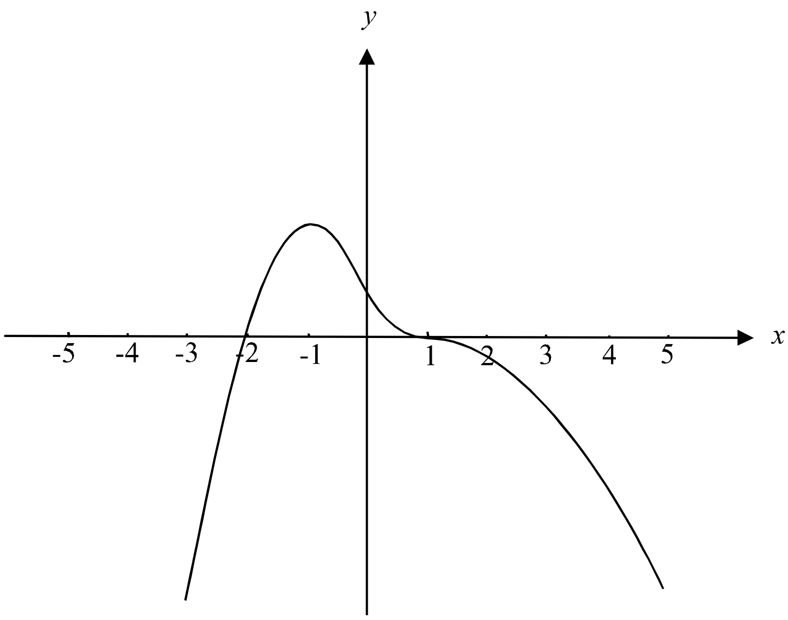
$$(t-4)(t+2) = 0$$

$t = 4$ or $t = -2$ but $t \in [0,10]$

so $t = 4$ **(1 mark)**

Question 5

a. A possible graph of *g* is shown below.



(1 mark) for correct shape
(1 mark) for correct intercepts

b. The inverse function $g^{-1}(x)$ does not exist since the function $g(x)$ is not 1:1.
(1 mark)

Question 6

a. At points of intersection,

$$3x^2 - x + 2 = 5x + 11$$ **(1 mark)**

$$3x^2 - 6x - 9 = 0$$

$$3(x^2 - 2x - 3) = 0$$

$$3(x-3)(x+1) = 0$$

$x = 3$ or $x = -1$ are the x-ordinates of the points of intersection.

(1 mark)

b. Do a quick sketch.

$$\text{Area} = \int_{-1}^{3} \{(5x+11) - (3x^2 - x + 2)\}dx$$

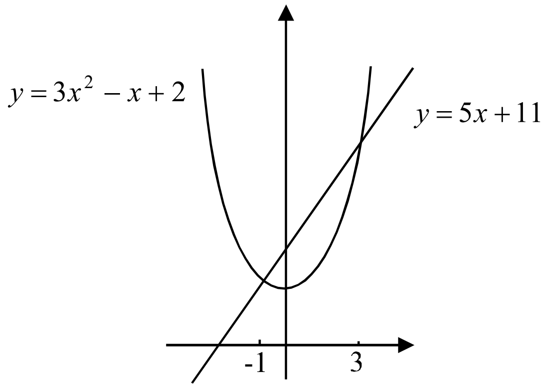
$$= \int_{-1}^{3} (-3x^2 + 6x + 9)dx$$ **(1 mark)**

$$= -3\int_{-1}^{3} (x^2 - 2x - 3)dx$$

$$= -3\left[\frac{x^3}{3} - \frac{2x^2}{2} - 3x\right]_{-1}^{3}$$ **(1 mark)**

$$= -3\left\{(9 - 9 - 9) - \left(-\frac{1}{3} - 1 + 3\right)\right\}$$

$$= 32 \text{ square units}$$



(1 mark)
Total 23 marks