THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 9836 5021
Fax 9836 5025

MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2004

1. E	8. B	15. D	22. D
2. C	9. D	16. A	23. C
3. C	10. E	17. A	24. D
4. A	11. E	18. C	25. C
5. A	12. D	19. B	26. E
6. B	13. A	20. A	27. D
7. D	14. C	21. A	

Part I – Multiple-choice solutions

Question 1

The graph crosses the x-axis at $x = a$ so we have an $x - a$ factor.
The graph touches the x-axis at $x = b$ so we have a repeated $x - b$ factor, i.e., $(x-b)^2$.
The cubic graph comes down from the left so the x^3 term must be negative.
The rule is

$$y = -(x-a)(x-b)^2$$
$$= (a-x)(x-b)^2$$

The answer is E.

Question 2

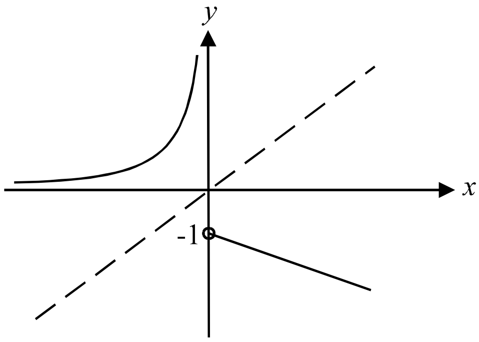
The graph of $y = e^x$ is reflected in the y-axis to become the graph of $y = e^{-x}$. This graph is then translated 3 units up to become $y = e^{-x} + 3$.
The answer is C.

Question 3

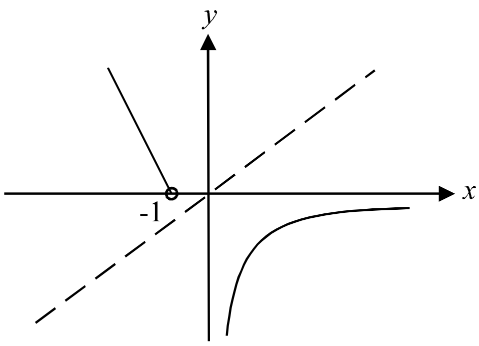
The graph of $y = e^{ax}$ has the asymptote $y = 0$.
The graph of $\log_e(ax)$ has the asymptote of $x = 0$.
The graph of $y = \sqrt{ax}$ does not have an asymptote.
The graph of $y = \frac{1}{ax}$ has asymptotes of $x = 0$ and $y = 0$.
The graph of $y = \frac{1}{ax^2}$ has asymptotes of $x = 0$ and $y = 0$.
The answer is C.

Question 4

To obtain the graph of an inverse function you reflect the original graph in the line $y = x$.
So,



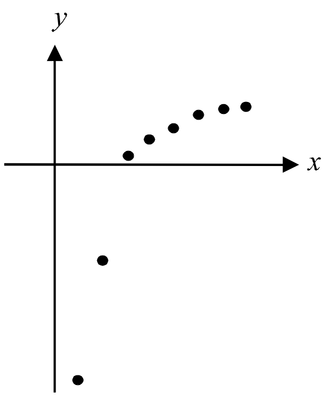
becomes



Note that the point $(0,-1)$ on the graph of $y = f(x)$ becomes $(-1, 0)$ on the graph of $y = f^{-1}(x)$.
The answer is A.

Question 5

Plot the data on your graphics calculator.



The data would be best modelled using a logarithmic function.
The answer is A.

Question 6

The shape of the graph is that of a sin graph that has been translated 1 unit down and has a period of π .

The graph of $y = \sin(nx)$ has a period of $\frac{2\pi}{n}$.

Since we require a period of π , we have $\pi = \frac{2\pi}{n}$ so $n = 2$.

The required equation is $y = \sin(2x) - 1$.

Note that an equation involving cos could be created but it is not one of the options offered here.

The answer is B.

Question 7

<u>Method 1</u>

Use your graphics calculator to sketch the graphs of $y = 0\cdot 3\tan\left(\frac{x}{2}\right)$ and $y = 1$ and find the point of intersection.

The x coordinate of this point of intersection is closest to 147°.

<u>Method 2</u>

Put your calculator in degree mode.

$$0\cdot 3\tan\left(\frac{x}{2}\right) = 1, \qquad x \in (-180°, 180°)$$

$$\tan\left(\frac{x}{2}\right) = 3\cdot\dot{3}$$

$$\frac{x}{2} = 73\cdot 3007...$$

$$x = 146\cdot 6015...$$

S	A
T	C

Note that the period of the graph of $y = 0\cdot 3\tan\left(\frac{x}{2}\right)$ is $\pi \div \frac{1}{2} = 2\pi$.

So there is only one solution to the equation $0\cdot 3\tan\left(\frac{x}{2}\right) = 1$.

Therefore the closest answer is 147°.

The answer is D.

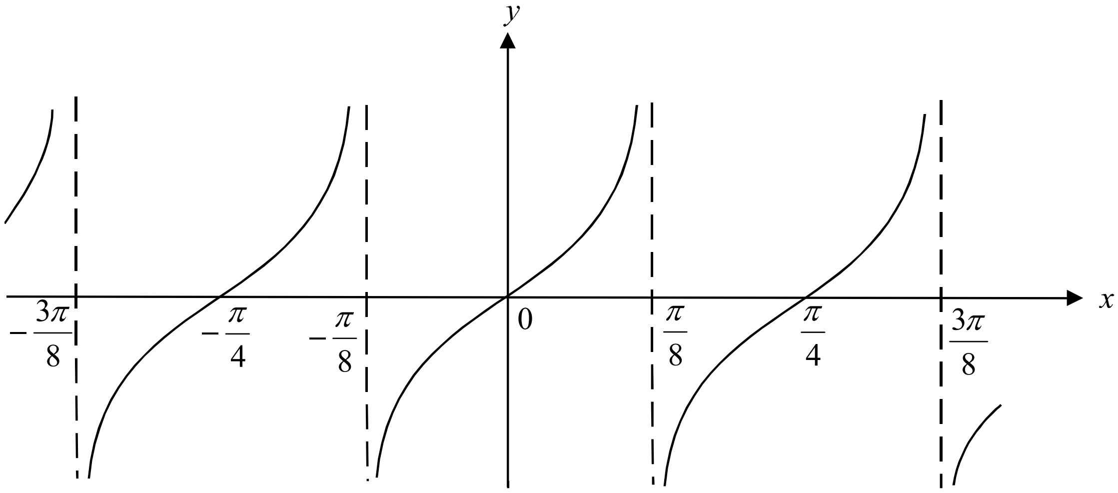
Question 8

Some key points to consider on the graph of $y = f(x)$ are the maximum and minimum turning points which occur at $x = -2\pi, 0, 2\pi$.
For these values of *x*, the corresponding value of *y* on the graph of $y = f'(x)$ will be zero.
Instantly we can eliminate options A and E.
The gradient of the graph of $y = f(x)$ is positive for $x \in (-2\pi, 0)$ and for values $x \in (2\pi, 4\pi)$ if we follow the pattern of the graph. Hence the graph of $y = f'(x)$ must be positive for these values of *x*. We can eliminate options C and D.
The answer is B.

Question 9

The period of the graph of $y = \tan(4x)$ is $\frac{\pi}{4}$.

Sketch the graph.



We see that over the domains $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $[-4\pi, \pi]$, $\left[0, \frac{\pi}{4}\right)$ the function is not defined because these domains include value(s) of *x* where an asymptote exists.

Only over the domain $x \in [0, \frac{\pi}{8})$ is the function defined.

The answer is D.

Question 10

$$\begin{aligned} y &= \sqrt{3x^2 - 4} \\ &= \left(3x^2 - 4\right)^{\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{1}{2}\left(3x^2 - 4\right)^{-\frac{1}{2}} \times 6x \qquad \text{(chain rule)} \\ &= \frac{3x}{\sqrt{3x^2 - 4}} \end{aligned}$$

The answer is E.

Question 11

Let $\quad y=\dfrac{\log_e(2x)}{x^3}$

So, $\quad \dfrac{dy}{dx}=\left(x^3\times\dfrac{2}{2x}-3x^2\log_e(2x)\right)\div x^6$

$$=\left(x^2-3x^2\log_e(2x)\right)\div x^6$$
$$=x^2\left(1-3\log_e(2x)\right)\div x^6$$
$$=\frac{1-3\log_e(2x)}{x^4}$$

The answer is E.

Question 12

Average rate of change between $t=0$ and $t=2$ is given by

$$\frac{f(2)-f(0)}{2-0}$$
$$=\frac{e^{\sqrt{4}}-e^0}{2}$$
$$=\frac{e^2-1}{2}$$

The answer is D.

Question 13

$$y=e^x\sin(x)$$
$$\frac{dy}{dx}=e^x\cos(x)+e^x\sin(x)$$

When $x=0$,

$$\frac{dy}{dx}=e^0\cos(0)+e^0\sin(0)$$
$$=1\times1+1\times0$$
$$=1$$

The gradient of the tangent is 1.

The gradient of the normal is $\dfrac{-1}{1}=-1$.

Now, when $x=0$,

$$y=e^0\sin(0)$$
$$=0$$

The equation of the normal to the curve at the point where $x=0$, is given by

$$y-0=-1(x-0)$$
$$y=-x$$

The answer is A.

Question 14

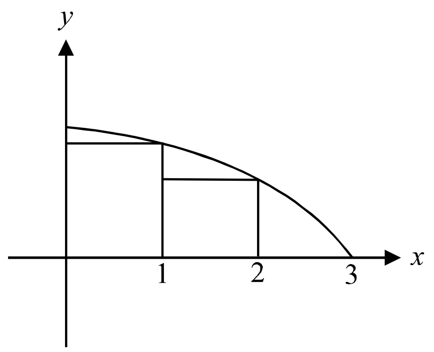
With $x = 4$, $f(x) = f(4)$
Now $x + h = 4 \cdot 01$
So $4 + h = 4 \cdot 01$
So $h = 0 \cdot 01$
So $f(x+h) \approx f(4) + 0 \cdot 01 f'(4)$
The answer is C.

Question 15

$y = g'(x)$ is the gradient function of the function *g*.
Hence the gradient of the graph of $y = g(x)$ will be positive when the graph of $y = g'(x)$ is positive. This occurs for $x \in (a, 0) \cup (d, f)$.
The answer is D.

Question 16

Sketch the graph and draw



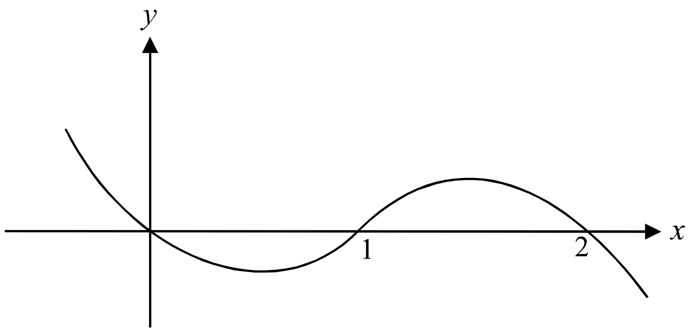
the "right" rectangles.
The required approximation is given by
$1 \times \log_e(4-1) + 1 \times \log_e(4-2)$
$= \log_e(3) + \log_e(2)$
$= \log_e(6)$
The answer is A.

Question 17

$$\frac{dy}{dx} = e^{3x} + \cos(3x)$$
$$y = \int \left(e^{3x} + \cos(3x)\right)dx$$
$$= \frac{1}{3}e^{3x} + \frac{1}{3}\sin(3x) + c$$
The answer is A.

Question 18

Sketch a graph which is easy to do because the cubic function is in factorised form.



Method 1

Use a graphics calculator (2^{nd} Calc $\int f(x)dx$) to find the two areas.

The answer is $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

Method 2

Now, $\quad -x(x-1)(x-2) = -x(x^2-3x+2)$

$$= -x^3+3x^2-2x$$

$$\text{Total area} = \int_1^2 (-x^3+3x^2-2x)dx - \int_0^1 (-x^3+3x^2-2x)dx$$

$$= \left[\frac{-x^4}{4}+\frac{3x^3}{3}-\frac{2x^2}{2}\right]_1^2 - \left[\frac{-x^4}{4}+\frac{3x^3}{3}-\frac{2x^2}{2}\right]_0^1$$

$$= \left\{(-4+8-4)-\left(-\frac{1}{4}+1-1\right)\right\}-\left\{\left(-\frac{1}{4}+1-1\right)-0\right\}$$

$$= \frac{1}{4}+\frac{1}{4}$$

$$= \frac{1}{2}$$

The answer is C.

Question 19

Now $\quad \int_1^2 g(x)dx = 5$

So $\quad \int_{-1}^2 (1-2g(x))dx = \int_{-1}^2 1\,dx - 2\int_{-1}^2 g(x)\,dx$

$$= [x]_{-1}^2 - 2\times 5$$

$$= 2--1-10$$

$$= -7$$

The answer is B.

Question 20

Method 1

The line of symmetry of a parabola occurs when $x = -\frac{b}{2a}$

$= \frac{-4}{2}$

$= -2$

So, $a \le -2$. The only value that a can be from the options is –5.

Method 2

Sketch the parabola, which is upright, on your graphics calculator and then locate the minimum of the parabola (2^{nd} Calc). This occurs at $(-2, 3)$.

So, $a \le -2$. The only value that a can be from the options is –5.

Method 3

$f(x) = x^2 + 4x + 7$

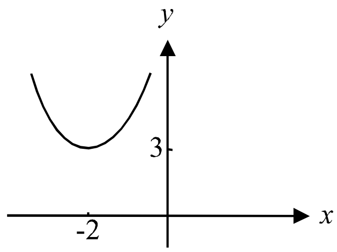
$= (x^2 + 4x + 4) - 4 + 7$ (completing the square)

$= (x+2)^2 + 3$ (turning point form)

We have an upright parabola with a turning point of $(-2, 3)$.

If the inverse function f^{-1} is to exist then f must be 1:1. So, $a \le -2$. The only value that a can be from the options is –5.

The answer is A.



Question 21

The coefficient of x^3 in the expansion of $(2x-1)^6$ is $20 \times 2^3 \times -1^3 = -160$.

The answer is A.

Question 22

$$\log_e\left(\frac{1}{x}\right) + 5\log_e(x) = \log_e(1)$$

$$\log_e\left(\frac{1}{x}\right) + \log_e(x^5) = \log_e(1)$$

$$\log_e\left(\frac{1}{x} \times x^5\right) = \log_e(1)$$

$$\log_e(x^4) = \log_e(1)$$

$$x^4 = 1$$

So $\quad x = \pm 1$

But, $x > 0$ since we have $\log_e(x)$ above.

So $x = 1$

The answer is D.

Question 23
The expected value of X is given by
$5 \times 0 \cdot 2 + 10 \times 0 \cdot 1 + 15 \times 0 \cdot 4 + 20 \times 0 \cdot 3 = 14$.
The answer is C.

Question 24
We have a binomial distribution where $p = 0 \cdot 85$, $n = 50$ and $x = 45$.
The required probability is given by

$$^{50}C_{45}(0 \cdot 85)^{45}(0 \cdot 15)^{5}$$

The answer is D.

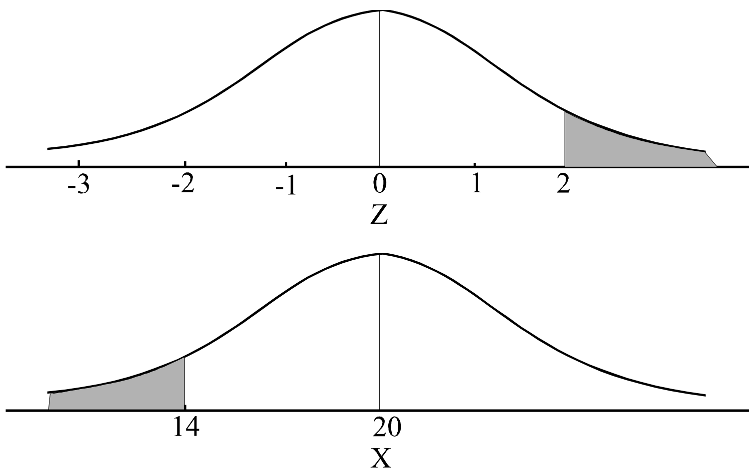
Question 25
In the class there are a total of 24 children. Six of these are in blue group and 18 are not in blue group. The principal selects without replacement so we have a hypergeometric distribution.
Where $D = 6$, $N = 24$, $x = 2$ and $n = 3$

So, $\Pr(X = 2) = \dfrac{^{6}C_{2}\ ^{18}C_{1}}{^{24}C_{3}}$

The answer is C.

Question 26
With 8 trials and a probability of success of 0.7 we would expect the graph to be negatively skewed.
If the probability of success was 0.5 the graph would be symmetrical.
Also $\mu = np$
$= 8 \times 0.7 = 5.6$ so the highest part of the graph should be between 5 and 6.
The only feasible graph is E.
The answer is E.

Question 27



If $\Pr(X < 14) = \Pr(z > 2)$, then because of the symmetry of the normal curve, 14 is two standard deviations from the mean.
So $20 - 14 = 6$
So one standard deviation is $6 \div 2 = 3$.
Note also that standard deviations can't be negative.
The answer is D.

PART II

Question 1

a. $e^x\left(e^{-x}-1\right)^2$

$=e^x\left(e^{-2x}-2e^{-x}+1\right)$

$=e^{-x}-2e^0+e^x$

$=e^{-x}-2+e^x$

(1 mark)

b. Let $P(x)=x^4+6x^3+ax^2-11x+3$

If $P(x)$ is exactly divisible by $(x+3)$, then $P(-3)=0$ **(1 mark)**

So, $P(-3)=81-162+9a+33+3=0$

$9a-45=0$

$a=5$

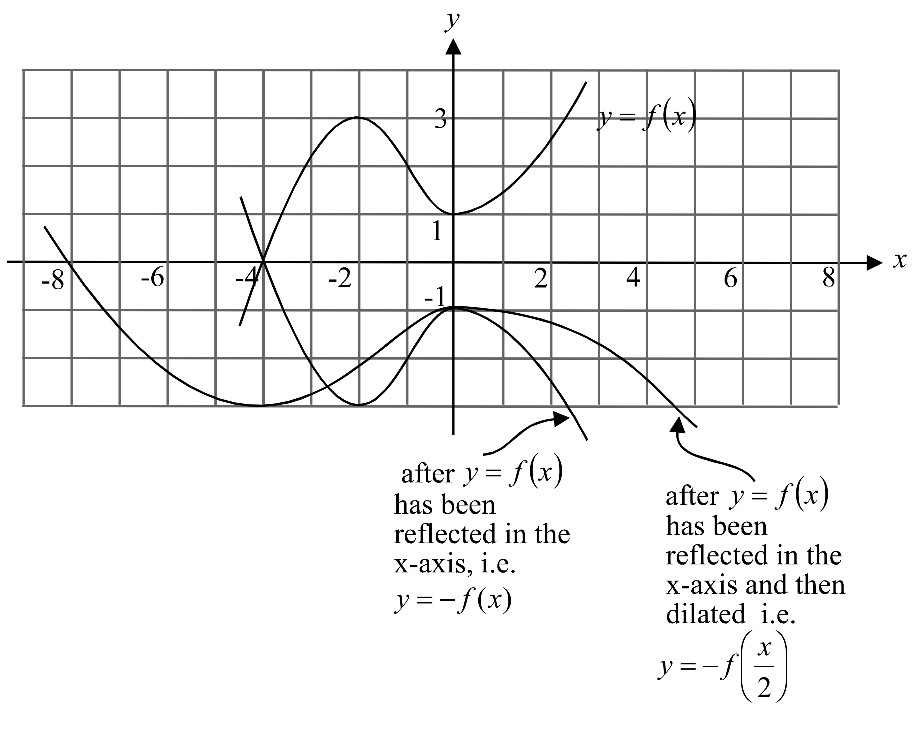
(1 mark)

Question 2

First sketch the graph of $y=f(x)$ after it has been reflected in the x-axis.

(1 mark) for graph of reflection

Then dilate this graph, i.e. stretch this graph parallel to the x-axis by a factor of two. This means that effectively each x-coordinate is multiplied by 2.



(1 mark) for graph of dilation

Question 3

a.

Number of sixes (X)	Probability $\Pr(X = x)$
0	$\frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$
1	$\frac{5}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{5}{6}=\frac{10}{36}$
2	$\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$

(2 marks)
(1 or 2 correct = **1 mark**, 3 correct = **2 marks**)

b. We have a binomial distribution where $n = 5$ and $p = 0\cdot 3$.

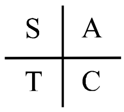
$$\begin{aligned}\Pr(X > 3) &= \Pr(X = 4) + \Pr(X = 5)\\ &= {}^5C_4(0\cdot 3)^4(0\cdot 7)^1 + {}^5C_5(0\cdot 3)^5(0\cdot 7)^0\\ &= 0\cdot 02835 + 0\cdot 00243\\ &= 0\cdot 0308 \quad \text{correct to 4 decimal places}\end{aligned}$$

(1 mark)

Question 4

a.

$$\begin{aligned} f(x) &= 0 & x &\in [0,\pi]\\ 2\cos(2x)+\sqrt{3} &= 0 & 2x &\in [0,2\pi]\\ \cos(2x) &= \frac{-\sqrt{3}}{2} \\ 2x &= \frac{5\pi}{6}, \frac{7\pi}{6}\\ x &= \frac{5\pi}{12}, \frac{7\pi}{12}\end{aligned}$$



(1 mark)

b.

$$f(x) = 2\cos(2x)+\sqrt{3}$$
$$f'(x) = -4\sin(2x)$$

(1 mark)

c. The range of the graph of $y = f'(x)$ is $-4 \le y \le 4$.
Hence $f'(x) \le 4$ for all x.

(1 mark)

Question 5

a. $\frac{d}{dx}\left(5x\log_e(2x)\right) = 5\log_e(2x) + 5x \times \frac{2}{2x}$

$= 5\log_e(2x) + 5$

(1 mark)

b. Method 1

From **a.**,

$5\log_e(2x) = \frac{d}{dx}\left(5x\log_e(2x)\right) - 5$

$\log_e(2x) = \frac{d}{dx}\left(x\log_e(2x)\right) - 1$

(1 mark)

$\int \log_e(2x)dx = \int \frac{d}{dx}\left(x\log_e(2x)\right)dx - \int 1dx$

$= x\log_e(2x) - x + c$ **(1 mark)**

Method 2

From **a.**,

$\int (5\log_e(2x) + 5)dx = 5x\log_e(2x)$ **(1 mark)**

$5\int \log_e(2x)dx + \int 5\,dx = 5x\log_e(2x)$

$5\int \log_e(2x)\,dx = 5x\log_e(2x) - \int 5\,dx$

$5\int \log_e(2x)\,dx = 5x\log_e(2x) - 5x + c$

$\int \log_e(2x)\,dx = x\log_e(2x) - x + c$ **(1 mark)**

Question 6

a. The maximal domain of $g(x)$

where $g(x) = \frac{1}{\sqrt{x-1}}$ is $x - 1 > 0$

$x > 1$

So $a = 1$.

(1 mark)

b. Method 1

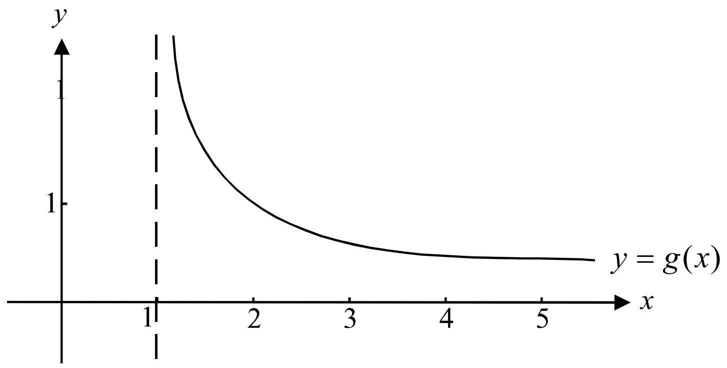
Find the x- intercept on the graph of $y = g(x)$.

We have $0 = \frac{1}{\sqrt{x-1}}$. Since $0 \neq 1$ there is no solution and hence there is no x-intercept on the graph of $y = g(x)$ Hence there is no y-intercept on the graph of $y = g^{-1}(x)$

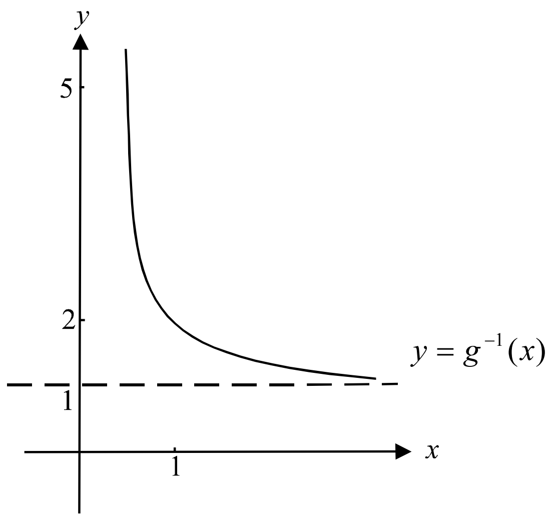
(1 mark)

<u>Method 2</u>

The graph of $y = g(x)$ is shown.



The graph of $y = g^{-1}(x)$ is a reflection of this graph in the line $y = x$.



Hence the graph of $y = g^{-1}(x)$ never crosses the *y*-axis.

(1 mark)

c. $$\int_n^5 g(x)\,dx = 2$$

So, $$\int_n^5 \frac{1}{\sqrt{x-1}}\,dx = 2$$

$$\int_n^5 (x-1)^{\frac{-1}{2}}\,dx = 2$$

$$\left[\frac{1}{\frac{-1}{2}+1}(x-1)^{\frac{1}{2}}\right]_n^5 = 2$$

(1 mark)

$$2\left\{4^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}\right\} = 2$$

$$2-(n-1)^{\frac{1}{2}} = 1$$

$$-(n-1)^{\frac{1}{2}} = -1$$

$$(n-1)^{\frac{1}{2}} = 1$$

$$n-1 = 1$$

$$n = 2$$

(1 mark)

Question 7

a. Find the equation of the tangent first.

Now, $y = \tan(2x)$

$$\frac{dy}{dx} = 2\sec^2(2x)$$

At $x = \frac{\pi}{8}$, $$\frac{dy}{dx} = 2\sec^2\left(\frac{\pi}{4}\right)$$

$$= \frac{2}{\cos^2\left(\frac{\pi}{4}\right)}$$

$$= \frac{2}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= 2 \div \frac{1}{2}$$

$$= 4$$

(1 mark)

The equation of the tangent is therefore

$$y-1=4\left(x-\frac{\pi}{8}\right)$$

$$y=4x-\frac{\pi}{2}+1$$

(1 mark)

This tangent crosses the x-axis when $y=0$.

So, $$0=4x-\frac{\pi}{2}+1$$

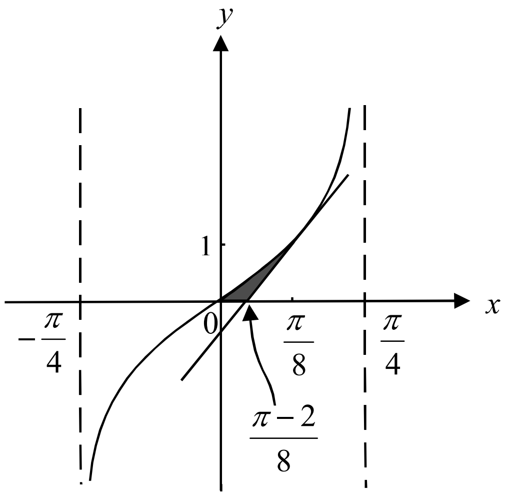
$$4x=\frac{\pi}{2}-1$$

$$4x=\frac{\pi-2}{2}$$

$$x=\frac{\pi-2}{8}$$ as required.

(1 mark)

b. Sketch a quick graph.



The area required is the shaded area.

$$\text{Area required}=\int_0^{\frac{\pi}{8}}\tan(2x)\,dx-\int_{\frac{\pi-2}{8}}^{\frac{\pi}{8}}(4x-\frac{\pi}{2}+1)dx$$

Alternatively,

$$\text{area required }=\int_0^{\frac{\pi-2}{8}}\tan(2x)dx-\int_{\frac{\pi-2}{8}}^{\frac{\pi}{8}}\left\{\tan(2x)-\left(4x-\frac{\pi}{2}+1\right)\right\}dx$$

(1 mark) first integral
(1 mark) second integral
Total 23 marks