2003 Mathematical Methods Written Examination 2 (analysis task) Suggested answers and solutions

Question 1

a Using the normal cdf distribution program on the graphics calculator gives:

normalcdf (141.5, ∞ , 140, 1.2) = 0.106 to three decimal places.

(Note: Be sure to write down what values were put into the calculator as any question worth two marks or more requires working to be shown, as well as the answer.)

b d cm either side of the mean indicates that area greater than $(140 + d)$ and less than $(140 - d)$ are both 0.075.

Invnorm (0.075, 140, 1.2) = 138.27

$\therefore \quad d = 140 - 138.27$

$= 1.7$ cm (correct to one decimal place).

c The Binomial distribution here is $(0.15 + 0.85)^{12}$ with the probability that a rod is faulty being 0.15.

$\Pr(X = 2) = {}^{12}C_2\,(0.15)^2(0.85)^{10}$

$= 0.292$ (correct to three decimal places).

d Hypergeometric distribution with N = 25, $n = 12$ and D = 4.

Probability (*at least* 2 rods) = Pr(2) + Pr(3) + Pr(4)

$$= \frac{{}^4C_2 \times {}^{21}C_{10}}{{}^{25}C_{12}} + \frac{{}^4C_3 \times {}^{21}C_9}{{}^{25}C_{12}} + \frac{{}^4C_4 \times {}^{21}C_8}{{}^{25}C_{12}}$$

$= 0.4070 + 0.2261 + 0.0391$

$= 0.672$ (correct to three decimal places).

e i $k = 1 - (0.15 + 0.17)$ which is 0.68

ii $E(y) = 0.68\,(x - 5) + 0 \times 0.15 + 0.17\,(x - 8)$

$= 0.17\,(5x - 28)$ or $0.85x - 4.76$.

iii If $E(y) = 0$ then $x = \$5.60$.

iv $\dfrac{0.68}{0.68 + 0.17} \times 100\% = 80\%$

Question 2

a Substitute $x = 4$ into $y = 2 - 2\cos\frac{x}{2}$ gives 2.83 m to two decimal places.

b $\dfrac{dy}{dx} = \sin\dfrac{x}{2}$

Now the maximum value that sine can have is 1 within the specified domain. Therefore the gradient must always be less than or equal to 1.

c The area under the curve $= \displaystyle\int_{-4}^{4} (2 - 2\cos\frac{x}{2})dx$

$= 2\displaystyle\int_{0}^{4} (2 - 2\cos\frac{x}{2})dx$

$= 2\left[\, 2x - 4\sin\frac{x}{2}\,\right]_0^4$

$= 2\,[\,(8 - 4\sin 2) - (0 - 0)]$

$= 8.73$ square metres (correct to two decimal places).

d i Find the x value where $y = 1$ as A is 1 metre above the x-axis.

$1 = 2 - 2\cos\frac{x}{2}$ and so $\cos\frac{x}{2} = \frac{1}{2}$

Therefore $\frac{x}{2} = \frac{\pi}{3}$ leading to $x = \frac{2\pi}{3}$, the required x co-ordinate.

ii To find the gradient of the normal at this value of x, find the gradient of the tangent (same as gradient of the curve) and then find the negative reciprocal.

The gradient of the tangent is:

$\dfrac{dy}{dx} = \sin\dfrac{x}{2}$ which is $\dfrac{\sqrt{3}}{2}$ at $x = \dfrac{2\pi}{3}$.

The gradient of the normal is $\dfrac{-2}{\sqrt{3}}$ or $\dfrac{-2\sqrt{3}}{3}$.

iii Before the length of AB can be found it is necessary to find the **equation** of the normal and then find where this cuts the x-axis by substituting $y = 0$.

Equation of AB : $y - 1 = \frac{-2\sqrt{3}}{3}\left(x - \frac{2\pi}{3}\right)$

At $y = 0$ $\quad x = \frac{\sqrt{3}}{2} + \frac{2\pi}{3}$.

The length of the straight line from $\left(\frac{2\pi}{3}, 1\right)$ to $\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}, 0\right)$ is obtained by substituting into $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. It gives a length of $\frac{\sqrt{7}}{2}$ metres .

Question 3

a $f(x) = x^3 e^{-2x}$

Using Product Rule

$f'(x) = 3x^2 e^{-2x} - 2x^3 e^{-2x}$

$= e^{-2x}(-2x^3 + 3x^2)$.

$= e^{-2x}(ax^3 + bx^2)$ if $a = -2$ and $b = 3$.

b Stationary values occur where $f'(x) = 0$.

If $e^{-2x}(-2x^3 + 3x^2) = 0$ then $-2x^3 + 3x^2 = 0$ as the exponential function e^{-2x} is never zero.

$x^2(3 - 2x) = 0$ and so $x = 0$ and $x = 1.5$

The stationary values are: (0, 0) stationary point of inflexion $(1.5, 3.375e^{-3})$ maximum turning point.

c i At $x = 1$ $y = e^{-2}$ and $f'(1) = e^{-2}(-2 + 3) = e^{-2}$

The equation of the tangent is :

$y - e^{-2} = e^{-2}(x - 1)$

Therefore $y - e^{-2} = e^{-2}x - e^{-2}$.

Hence $y = e^{-2}x$ as required.

ii At $x = 0$ $f'(0) = 0$ and so the equation is :

$y = 0$

iii Choose any point (p, p^3e^{-2p}) on the curve and find the equation of the tangent at that point.

$y - p^3e^{-2p} = e^{-2p}(-2p^3 + 3p^2)(x - p)$

$y - e^{-2p}(-2p^3 + 3p^2)x$

$= p^3e^{-2p} - p\,e^{-2p}(-2p^3 + 3p^2)$

This tangent will pass through the origin only if its y-intercept is zero, that is, the RHS of this equation is zero.

Therefore $\quad 0 = p^3e^{-2p} - p\,e^{-2p}(-2p^3 + 3p^2)$

$= p^3e^{-2p} + 2p^4e^{-2p} - 3p^3e^{-2p}$

$= 2p^4e^{-2p} - 2p^3e^{-2p}$

$= 2p^3e^{-2p}(p - 1)$

Now this occurs if $p = 0$ or $p = 1$ only as e^{-2p} is never zero. Therefore these are the only two tangents that pass through the origin.

d i Using the Product Rule:

$\frac{d}{dx}(4x^3 + px^2 + qx + 3)e^{-2x}$

$= (12x^2 + 2px + q)e^{-2x} - 2(4x^3 + px^2 + qx + 3)e^{-2x}$

$= e^{-2x}[-8x^3 + x^2(12 - 2p) + x(2p - 2q) + (q - 6)]$

But the derivative given only has an x^3e^{-2x} term.

Equate the coefficients of x^2 to zero:

$12 - 2p = 0$ and so $p = 6$

Equate the coefficients of x to zero:

$2p - 2q = 0$ and so $p = q$

and the constant coefficient: $q - 6 = 0$, $q = 6$

The coefficients of x^3 give $k = -8$.

ii The area between the tangent and curve between $x = 0$ and $x = 1$ is given by

$$\int_0^1 (e^{-2}x - x^3 e^{-2x})dx$$

$$= \int_0^1 (e^{-2}x)dx - \int_0^1 (x^3 e^{-2x})dx$$

$$= \int_0^1 (e^{-2}x)dx + \frac{1}{8}\int_0^1 (-8x^3 e^{-2x})dx$$

$$= \int_0^1 (e^{-2}x)dx + \frac{1}{8}\int_0^1 \frac{d}{dx}(4x^3 + 6x^2 + 6x + 3)\, e^{-2x}\, dx$$

$$= [\frac{1}{2}e^{-2}x^2]_0^1 + \frac{1}{8}[(4x^3 + 6x^2 + 6x + 3)\, e^{-2x}]_0^1$$

$$= \left(\frac{1}{2}e^{-2} - 0\right) + \frac{1}{8}\left[\left(4 + 6 + 6 + 3\right)e^{-2} - 3e^0\right]$$

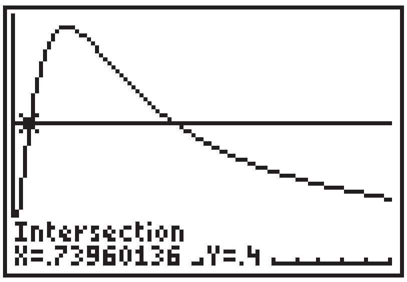
$$= \frac{1}{2}e^{-2} + \frac{1}{8}\left(19e^{-2} - 3\right)$$

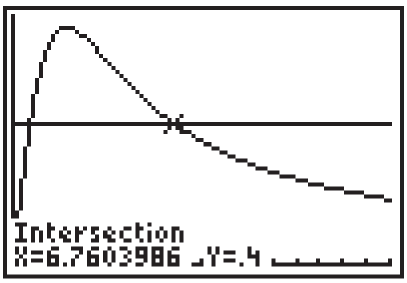
$$= \frac{23}{8}e^{-2} - \frac{3}{8}$$

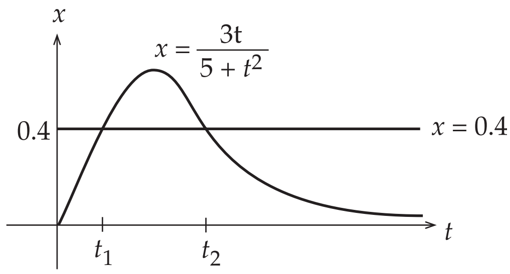
Question 4

a $\frac{3t}{5 + t^2} \geq 0.4$ needs to be solved.

Graph $y_1 = \frac{3t}{5 + t^2}$ and $y_2 = 0.4$ and find their intersection for $0 \leq t \leq 16$ and $0 \leq x \leq 0.7$.







The number of hours
$= 6.7603986 - 0.73960136$ which is 6.02 to two decimal places.

b $\frac{dx}{dt} = \frac{(5 + t^2)3 - 3t.2t}{(5 + t^2)^2}$

$$= \frac{15 + 3t^2 - 6t^2}{(5 + t^2)^2}$$

$= 0$ for a turning point.

This occurs if $3t^2 = 15$ and so $t = \sqrt{5}$, as $t \geq 0$.

Substituting $t = \sqrt{5}$ into x gives an exact value of $x = \frac{3\sqrt{5}}{10}$.

c Graphing calculator:

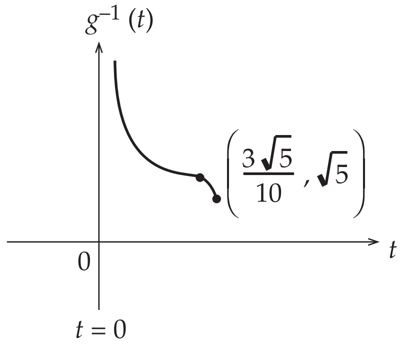
Graph $y_1 = \frac{15 - 3t^2}{(5 + t^2)^2}$

and $y_2 = 0.25$
and find their intersection.
$t = 1.22$ hours
(Correct to two decimal places).

d i The function must be one-to-one to have an inverse. Since the upper values tend to infinity, the lower restriction on t will occur at the maximum value. $a = \sqrt{5}$

ii The end-point **must** be shown as $\left(\frac{3\sqrt{5}}{10}, \sqrt{5}\right)$.

The equation $t = 0$ of the vertical asymptote **must** also be clearly put on the graph.
The concavity of the end-point is important here.



iii Switch x and t in $x = \frac{3t}{5+t^2}$ to find the inverse function.

Therefore $t = \frac{3x}{5+x^2}$.

$$t(5 + x^2) = 3x$$

$$x^2t + 5t = 3x$$

$$x^2t - 3x + 5t = 0$$

Solving this quadratic gives:

$$x = \frac{3 \pm \sqrt{9 - 20t^2}}{2t}$$

Now the negative square root sign is **not** the inverse for the specified domain as

$r_{g^{-1}} = d_g$

Therefore $g^{-1}(x) = \frac{3 + \sqrt{9 - 20t^2}}{2t}$

