THE HEFFERNAN GROUP

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MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 2003 SOLUTIONS

Question 1

a. We need to find the value of t where the graph of $y = r(t)$ intersects with the t-axis since on the t-axis, $r = 0$, that is, the rate of rainfall is zero ml/hr.
From the graph, we see that this happens between $t = 3$ and $t = 4$ secs. Using a graphics calculator, sketch the graph of $y = r(t)$ and find the t-intercept.
The value of t is 3.259921. **(1 mark)**
The question asks for the time to the nearest minute. We have 3 hours and 0.259921 hours, which is $0 \cdot 259921 \times 60$ minutes. So the rain lasts for the 3 hours and 16 minutes (to the nearest minute). **(1 mark)**

b. <u>Method 1</u> – algebraic

Now, $r(t) = -t^4 + 6t^3 - 12t^2 + 10t$

So, $r'(t) = -4t^3 + 18t^2 - 24t + 10$ **(1 mark)**

Try , $r'(1) = -4 + 18 - 24 + 10 = 0$

so, $(t-1)$ is a factor

$$\begin{aligned} r'(t) &= -4t^2(t-1) + 14t(t-1) - 10(t-1) \\ &= (t-1)(-4t^2 + 14t - 10) \quad \textbf{(1 mark)} \\ &= (t-1)(-2t+2)(2t-5) \\ &= -4(t-1)(t-1)(t-2\cdot 5) \\ &= -4(t-1)^2(t-2\cdot 5) \quad \textbf{(1 mark)} \end{aligned}$$

as required

<u>Method 2</u> – using a calculator

Now, $r(t) = -t^4 + 6t^3 - 12t^2 + 10t$

So, $r'(t) = -4t^3 + 18t^2 - 24t + 10$ **(1 mark)**

Sketch the graph of $y = r'(t)$.
Note that $r'(t) = 0$ when $t = 1$ and $t = 2 \cdot 5$ and at $t = 1$ the graph touches but does not cut, suggesting a repeated root and hence a repeated factor of $(t-1)^2$.
So $r'(t) = a(t-1)^2(t - 2\cdot 5)$ **(1 mark)**
where a is a constant which must be negative since the graph is coming down from the left. To obtain a constant term of 10, a must be – 4.
So $r'(t) = -4(t-1)^2(t-2\cdot 5)$. **(1 mark)**

c. We want to find

$r'(t) = 0$

From part b.,

$r'(t) = -4(t-1)^2(t-2\cdot 5) = 0$

$t = 1 \quad \text{or } t = 2\cdot 5$

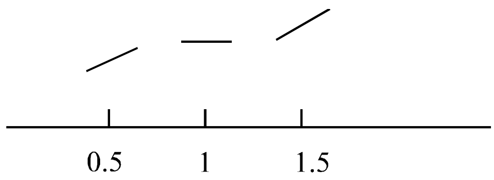
From the graph, we know that the maximum occurs between t = 2 and t = 3. So the maximum rate must occur at $t = 2.5$. **(1 mark)**

So $\quad r(2\cdot 5) = 4\cdot 6875 \text{ ml/hr}$. **(1 mark)**

d. At a stationary point of inflection $r'(t) = 0$. From part b., we know that the stationary points occur at $t = 1$ or $t = 2\cdot 5$. There is no stationary point at $t = 2$ and hence the point of inflection is not a stationary point of inflection.

(1 mark)

e. **i.** Let us investigate around $t = 1$. We know that $r'(1) = 0$.



$r'(0\cdot 5) = -4 \times 0.25 \times -2 = +ve$

$r'(1\cdot 5) = -4 \times 0.25 \times -1 = +ve$

The gradient is positive on either side of the stationary point.
So at $t = 1$, there is a stationary point of inflection. **(1 mark)**

ii. $r'(t)$ is the gradient function. The gradient of the function $y = r(t)$ is positive for $t \in (0, 1) \cup (1, 2\cdot 5)$. **(1 mark)**

Note that at t = 1 and at $t = 2\cdot 5$ the gradient of $y = r(t)$ is zero (from parts **e. i.** and **c.**)

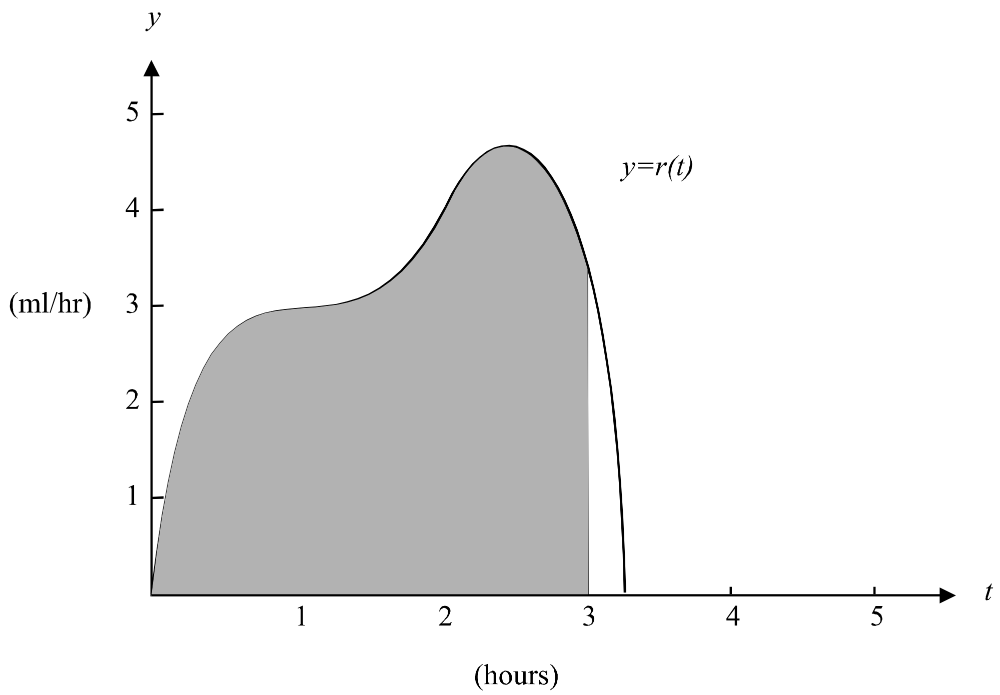
f. On a graphics calculator, sketch the graph of $y = r(t)$ and the graph of $y = 2$. Find the two points of intersection. One is $(0\cdot 2833, 2)$ and the other is $(3\cdot 1069, 2)$.

(1 mark)

So the difference in the time coordinates is $2\cdot 8$ hours (correct to 1 decimal place).

(1 mark)

g. The amount of rain to fall during the first three hours of this episode is equal to the area of the shaded region below.



$$\text{Rainfall} = \int_0^3 r(t)\,dt$$

$$= \int_0^3 \left(-t^4 + 6t^3 - 12t^2 + 10t\right)dt \qquad \textbf{(1 mark)}$$

$$= \left[\frac{-t^5}{5} + \frac{6t^4}{4} - \frac{12t^3}{3} + \frac{10t^2}{2}\right]_0^3$$

$$= \left[\frac{-t^5}{5} + \frac{3t^4}{2} - 4t^3 + 5t^2\right]_0^3$$

$$= (-48\cdot 6 + 121\cdot 5 - 108 + 45) - (0)$$

$$= 9\cdot 9$$

So 9.9 ml of rain fell.

(1 mark)
Total 14 marks

Question 2

a. **i.** We have a binomial variable with $n = 5$ and $p = 0 \cdot 7$ (since success is being randomly assigned a bay coloured horse).

So the expected number $= 5 \times 0 \cdot 7$

$= 3 \cdot 5$

(1 mark)

ii. The number of people in Sue's group who are randomly assigned a bay coloured horse is a binomial variable with $n = 5, p = 0 \cdot 7$ and $x = 2$.

So, $\Pr(x = 2) = {}^5C_2 (0 \cdot 7)^2 (0 \cdot 3)^3$ **(1 mark)**

$= 0 \cdot 1323$ (correct to 4 decimal places)

(1 mark)

iii. Pr(at least one person gets a bay horse) $= \Pr(x = 1) + \Pr(x = 2) + \Pr(x = 4) + \Pr(x = 5)$

$= 1 - \Pr(x = 0)$ **(1 mark)**

$= 1 - {}^5C_0 (0 \cdot 7)^0 (0 \cdot 3)^5$

$= 1 - 0 \cdot 00243$

$= 0 \cdot 9976$ (correct to 4 decimal places)

(1 mark)

b. **i.**

$\Pr(X > 160) = \Pr(Z > 1 \cdot 25)$ since $z = \dfrac{160 - 150}{8}$

$= 1 - \Pr(Z < 1 \cdot 25)$ $\quad = 1 \cdot 25$

$= 1 - 0 \cdot 8944$

$= 0 \cdot 1056$

as required

(1 mark)

ii. Each member of the group is randomly assigned a horse and we know from part **i.** that the probability that Sue is randomly assigned a horse of height greater than 160cm is $0 \cdot 1056$. The probability that each member gets a horse with height greater than 160cm is

$= 0 \cdot 1056^5$

$= 0 \cdot 00001$

(1 mark)

c. This follows a binomial distribution with $n = 5, x = 2$ and $p = 0 \cdot 1056$.

(1 mark) for recognition of a binomial distribution

$\Pr(x = 2) = {}^5C_2 (0 \cdot 1056)^2 (0 \cdot 8944)^3$ **(1 mark)** for correct substitution

$= 0 \cdot 0798$ (correct to 4 decimal places) **(1 mark)** for correct answer

d. Method 1 – Use the hypergeometric formula.

When randomly choosing who will cross the bridge, there is no replacement since once someone has crossed, that is it, they can't be chosen again.
So we have a hypergeometric distribution with $n = 3,\ x = 2,\ D = 2$ and $N = 5$.

$$\Pr(X = 2) = \frac{{}^2C_2\ {}^3C_1}{{}^5C_3}$$ **(1 mark)** recognition of hypergeometric

$$= \frac{3}{10}$$ **(1 mark)** for correct answer

Method 2 – Outline all the possibilities.

The possible orders in which they might cross where two males are amongst the first three are

MMFFF or MFMFF or FMMFF **(1 mark)**

We have $\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}\times\frac{2}{2}\times\frac{1}{1}+\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}\times\frac{2}{2}\times\frac{1}{1}+\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}\times\frac{1}{1}$

$$= \frac{6}{60}+\frac{6}{60}+\frac{6}{60}$$

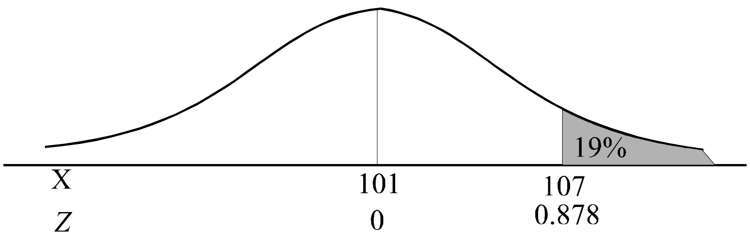
$$= \frac{3}{10}$$ **(1 mark)**

e. Since the height of the Shetland ponies is normally distributed, the distribution is symmetrical and if 19% of the ponies are less than 95cm high and 19% are greater than 107cm high, then the mean lies halfway in between these heights. So the mean is 101cm.

(1 mark)

Since 19% of the ponies are greater than 107cm high, then 81% are less than 107cm high.

(1 mark)



So, looking up the tables backwards or using your calculator, we find that $Z = 0\cdot 878$.

(1 mark)

Now $$Z = \frac{X - \mu}{\sigma}$$

$$0\cdot 878 = \frac{107 - 101}{\sigma}$$

So $\sigma = 6.83$

(1 mark)

Total 16 marks

Question 3

a. When $t = 0$, the see-saw is horizontal and the height of the see-saw above the ground is given by

$$h_E = 60\sin(0) + 60$$
$$= 60$$

So the height of the metal support is 60 cm. **(1 mark)**

b. We have a sine function which has an amplitude of 60 cm and which has been translated upwards by 60 cm. So, the maximum height reached is 120 cm. **(1 mark)**

c. The see-saw is horizontal again after Ella has reached the maximum height but before Meg has reached the maximum height. This is halfway through the period of one complete cycle of the see-saw. Now period $= \frac{2\pi}{n}$ where $n = \frac{\pi}{2}$ in this case.

So the period $= \frac{2\pi}{\pi} \times 2 = 4$ seconds. So halfway through this is 2 seconds which is when the see-saw is next horizontal. **(1 mark)**

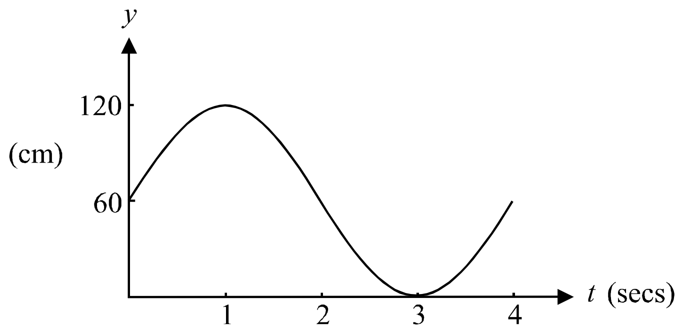
d. The see-sawing finishes when $t = 41 \cdot 5$.

Now, $h_E(41 \cdot 5) = 60\sin\left(\frac{41.5\pi}{2}\right) + 60$

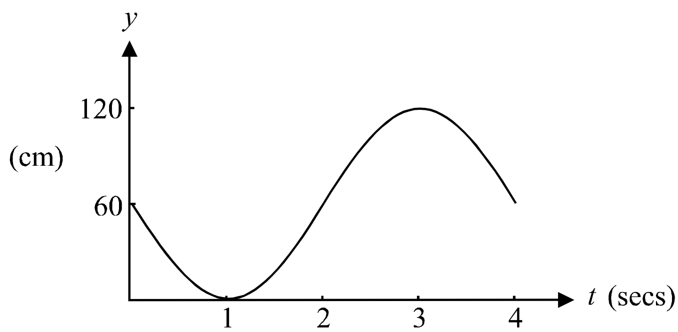
$$= 102 \cdot 4264$$

So Ella is $102 \cdot 43$ cm (to 2 decimal places) off the ground. **(1 mark)**

e. The graph of $y = h_E$ is shown below over 1 period.



Therefore the graph of $y = h_M$ must look like the graph below.



The graph is a reflection in the line $y = 60$.

A possible equation is $h_M = -60\sin\left(\frac{\pi t}{2}\right) + 60$.

(1 mark) for inclusion of a negative sign before the sine function

(1 mark) for correct inclusion of $\frac{\pi t}{2}$ and 60

Another alternative is $h_M = 60\sin\frac{\pi}{2}(t-2) + 60$ which is a sine function that has undergone a horizontal translation.

f. Ella's end of the see-saw is exactly 90cm off the ground when $h_E = 90$.
That is, when

$$60\sin\left(\frac{\pi t}{2}\right) + 60 = 90$$

$$60\sin\left(\frac{\pi t}{2}\right) = 30$$

$$\sin\left(\frac{\pi t}{2}\right) = \frac{1}{2}$$ **(1 mark)**

S	A
T	C

$$\frac{\pi t}{2} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \ldots$$

$$t = \frac{1}{3}, \frac{5}{3}, \frac{13}{3}, \frac{17}{3}, \frac{25}{3}, \frac{29}{3}, \frac{37}{3}, \ldots$$ **(1 mark)**

Now during the first ten seconds, the times when Ella's end of the see-saw is exactly 90cm off the ground are $t = \frac{1}{3}, 1\frac{2}{3}, 4\frac{1}{3}, 5\frac{2}{3}, 8\frac{1}{3}$ and $9\frac{2}{3}$.

(1 mark) for <u>all</u> the correct answers

g. **i.** The rate at which h_E changes with respect to time is given by $\frac{dh_E}{dt}$.

Now, $h_E = 60\sin\left(\frac{\pi t}{2}\right) + 60$

So, $\frac{dh_E}{dt} = 60 \times \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$

$= 30\pi\cos\left(\frac{\pi t}{2}\right)$

(1 mark)

ii. Now 1 metre/second = 100cm/second.
<u>Method 1</u>
Sketch the graph of $y = \frac{dh_E}{dt}$.
Notice that the maximum value of the function is approximately 94. **(1 mark)**
$\frac{dh_E}{dt}$ never reaches 100 and so Ella doesn't feel sick. **(1 mark)**

Method 2

Observe the function

$$\frac{dh_E}{dt} = 30\pi \cos\left(\frac{\pi t}{2}\right).$$

Note that the amplitude is $30\pi(\approx 94 \cdot 2)$ and there is no vertical translation.

So, the maximum value of the function is approximately 94. **(1 mark)**

So, $\frac{dh_E}{dt}$ never reaches 100 and so Ella doesn't feel sick. **(1 mark)**

Method 3

Algebraically

$$\frac{dh_E}{dt} = 30\pi \cos\left(\frac{\pi t}{2}\right) = 100$$

$$\cos\left(\frac{\pi t}{2}\right) = \frac{100}{30\pi}$$

$$\cos\left(\frac{\pi t}{2}\right) = 1.06$$ **(1 mark)**

Now, $$\frac{\pi t}{2} = \cos^{-1} 1.06$$
$$= \text{undefined}$$

So there is no point of intersection. So Ella never feels sick. **(1 mark)**

Total 12 marks

Question 4

a. Now "a" is the x-intercept. So we need to find the x-intercept.

Let $y = \log_e(x-1) + 2$

When $y = 0$, we have

$$0 = \log_e(x-1) + 2$$
$$-2 = \log_e(x-1)$$
$$e^{-2} = x - 1$$
$$x = 1 + e^{-2}$$

So $a = 1 + e^{-2}$ as required **(1 mark)**

b. **i.** $f(x) = \log_e(x-1) + 2$

Let $y = \log_e(x-1) + 2$

Swap x and y

$$x = \log_e(y-1) + 2$$

Rearrange

$$x - 2 = \log_e(y-1)$$
$$e^{x-2} = y - 1$$
$$y = 1 + e^{x-2}$$

So $f^{-1}(x) = 1 + e^{x-2}$ as required

(1 mark)

ii. $d_{f^{-1}} = r_f$

Now, $f(4) = \log_e(4-1) + 2$

$= \log_e(3) + 2$

So, $r_f = [0, \log_e(3) + 2]$

$d_{f^{-1}} = [0, \log_e(3) + 2]$ **(1 mark)**

iii. $r_{f^{-1}} = d_f$

Now $d_f = [a, 4]$

$= \left[1 + e^{-2}, 4\right]$

So, $r_{f^{-1}} = \left[1 + e^{-2}, 4\right]$

(1 mark)

c. $f(x) = \log_e(x-1) + 2$

$f'(x) = \dfrac{1}{x-1}$

$f'(2) = \dfrac{1}{1}$

$= 1$

So the gradient of the tangent at $x = 2$ is 1. **(1 mark)**

The equation of the tangent is given by $y - y_1 = m(x - x_1)$. If $x = 2$, then

$f(2) = \log_e 1 + 2$

$= 2$

So $(x_1, y_1) = (2,2)$ and $m = 1$

The required equation is

$y - 2 = 1(x - 2)$

$y = x$

(1 mark)

d. The point of contact is located at $(2,2)$. **(1 mark)**

Since the functions represent the outer edges of the cables, they can touch but not cross. Hence, they can only touch when they are on the line $y = x$ since they are inverse functions. From part **c.**, the equation of the tangent at $x = 2$ is $y = x$.

(1 mark)

e. $\int_0^2 \left(1 + e^{x-2}\right)dx$

Note that e^{-2} is a constant.
$e^{-2} \approx 0 \cdot 135$

$= \int_0^2 \left(1 + e^{-2}e^x\right)dx$

$= \left[x + e^{-2}e^x\right]_0^2$ **(1 mark)**

$= \left\{\left(2 + e^{-2}e^2\right) - \left(0 + e^{-2}e^0\right)\right\}$

$= 2 + e^0 - e^{-2}$

$= 3 - \dfrac{1}{e^2}$ as required. **(1 mark)**

Alternatively, since $\int e^{f(x)}dx = \frac{1}{f'(x)}e^{f(x)}$,

Then $\int_0^2 (1+e^{x-2})dx$

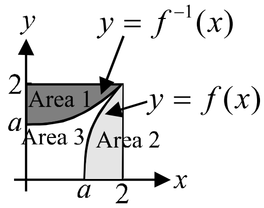
$= \left[x + e^{x-2}\right]_0^2$

$= (2+e^0) - (0+e^{-2})$

$= 3 - e^{-2}$

f. Note that $\int_a^2 f(x)dx = \int_a^2 (\log_e(x-1)+2)dx$. This integral is beyond the scope of the Maths Methods Unit 3 & 4 course so we need to employ an alternative strategy.

In the diagram, areas 1 and 2 are equal since the graphs of $y = f^{-1}(x)$ and $y = f(x)$ are symmetrical about the line $y = x$.
In part **e.** we found the sum of areas 2 and 3.
Now, the area of the square enclosing the 3 areas is $2\times 2 = 4$ square units.



So, $\text{area } 1 = 4 - (\text{area } 2 + \text{area } 3)$

$= 4 - \left(3 - \frac{1}{e^2}\right)$ from part e.

$= 1 + \frac{1}{e^2}$ **(1 mark)**

Since area $1 =$ area 2,

$\text{area } 3 = \text{area of square} - \text{area } 1 - \text{area } 2$

$= 2\times 2 - \left(1+\frac{1}{e^2}\right) - \left(1+\frac{1}{e^2}\right)$

$= 4 - 1 - \frac{1}{e^2} - 1 - \frac{1}{e^2}$

$= 2 - \frac{2}{e^2}$ **(1 mark)**

Now, because of the symmetry of the position of the 4 cables inside the cabling, the total cavity is given by

$4\left(2 - \frac{2}{e^2}\right)$

$= 8\left(1 - \frac{1}{e^2}\right)$ square units

(1 mark)
Total 13 marks