THE
HEFFERNAN
GROUP

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MATHS METHODS 3 & 4
TRIAL EXAMINATION 2
2002
SOLUTIONS

Question 1

a.

X	0	1	2	3
$\Pr(X = x)$	0.4	0.3	0.2	0.1

(1 mark)

b. The expected number of sachets of sugar a customer uses in their coffee is given by

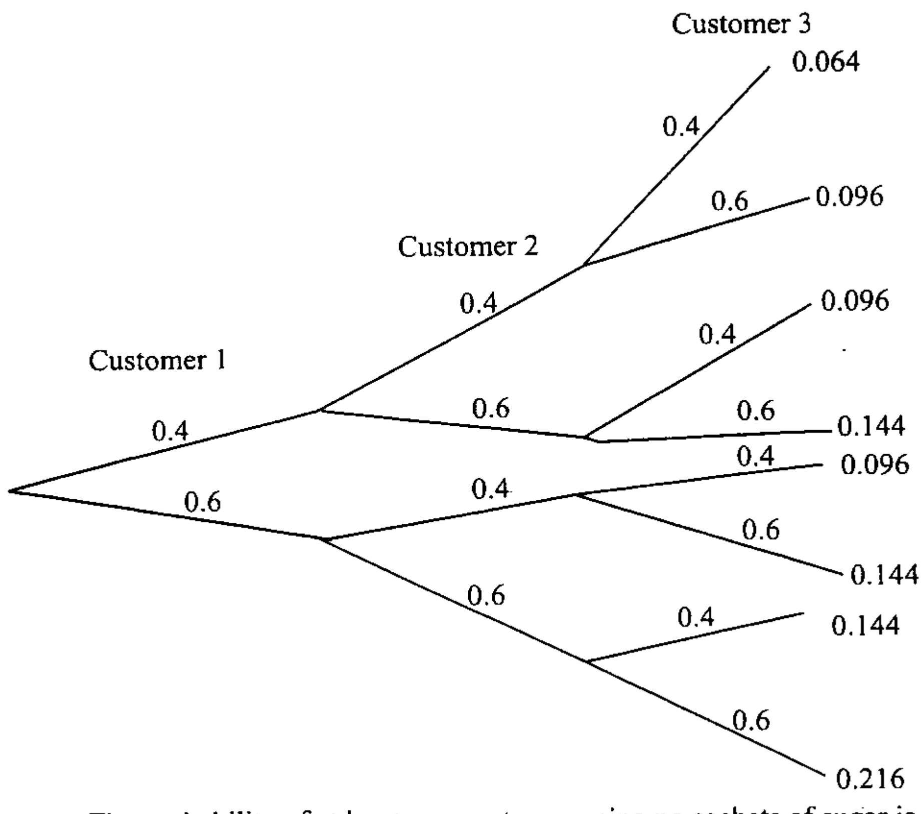
$0 \times 0.4 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.1 = 1$ **(1 mark)**

So the expected number of sachets of sugar used by the next 100 customers who buy a coffee is $100 \times 1 = 100$. **(1 mark)**

c. **i.** The probability that a customer uses 1 sachet of sugar is 0.3. The probability that 3 customers use 1 sachet of sugar is $0.3 \times 0.3 \times 0.3 = 0.027$. **(1 mark)**

ii. Method 1

Draw a tree diagram. The probability of a customer using no sachets is 0.4.
The probability of a customer using 1 or more sachets is 0.6. **(1 mark)**



The probability of at least one customer using no sachets of sugar is
$0.064 + 0.096 + 0.096 + 0.144 + 0.096 + 0.144 + 0.144 = 0.784$. **(1 mark)**

Method 2

$\Pr(\text{at least one customer uses no sachets of sugar})$

$= 1 - \Pr(\text{no one uses no sachets})$ **(1 mark)**

$= 1 - {}^3C_0 (0.4)^0 (0.6)^3$

$= 1 - 1 \times 1 \times 0.6^3$

$= 0.784$ **(1 mark)**

iii. List the ways that between the 3 customers, a total of 2 sachets of sugar can be used.

Outcome	Customer A	Customer B	Customer C	Probability
1	1	1	0	0.036
2	1	0	1	0.036
3	0	1	1	0.036
4	2	0	0	0.032
5	0	2	0	0.032
6	0	0	2	0.032
			Total	0.204

The total probability for these six (favourable) outcomes is 0.204.

(1 mark) for possible outcomes
(1 mark) for probability of each outcome
(1 mark) for total probability

d. **i.** $\Pr(Y < 5)$ where Y is the weight of a sachet of sugar and Y is distributed normally.

$$\text{Now, } Z = \frac{5-5.2}{0.1} = -2$$

Method 1 Using tables

$$\Pr(Z < -2) = 1 - \Pr(Z < 2) = 1 - 0.9772 \quad \text{from the tables}$$
$$= 0.0228 \quad \text{(to 4 decimal paces)}$$

(1 mark)

Method 2 Using a graphics calculator

$$\text{normalcdf}(0,5,5.2,0.1) = 0.02275 = 0.0228 \text{ (to 4 decimal places)}$$

So the probability that the next customer who uses a sachet of sugar uses one which weighs less than 5 grams is 0.0228 (to 4 decimal places)

(1 mark)

ii. From part **i.** the probability of choosing a sachet of sugar which weighs less than 5 g is 0.0228(to 4 decimal places).
Sue can use a sachet of sugar weighing less than 5 g by using 1 sachet of sugar and then choosing one that weighs less than 5 g OR by using 2 sachets of sugar and then choosing one that weighs less than 5 g OR by using 3 sachets of sugar and then choosing one that weighs less than 5 g.

(1 mark)

Required probability

$$= 0.3 \times 0.0228 + 0.2 \times {}^2C_1 (0.0228)^1 (0.9772)^1 + 0.1 \times {}^3C_1 (0.0228)^1 (0.9772)^2$$

(1 mark)

$$= 0.00684 + 0.008912 + 0.00653165$$
$$= 0.0223 \text{ (correct to 4 decimal places)}$$

(1 mark)
Total 13 marks

Question 2

a. **i.** $y = g(x)$

$y = \dfrac{1}{x-a} + a$

when $x = 0$, $y = -\dfrac{1}{a} + a$ which is the y-intercept **(1 mark)**

when $y = 0$, $0 = \dfrac{1}{x-a} + a$

$-a = \dfrac{1}{x-a}$

$-a(x-a) = 1$

$x - a = -\dfrac{1}{a}$

$x = \dfrac{-1}{a} + a$ which is the x - intercept

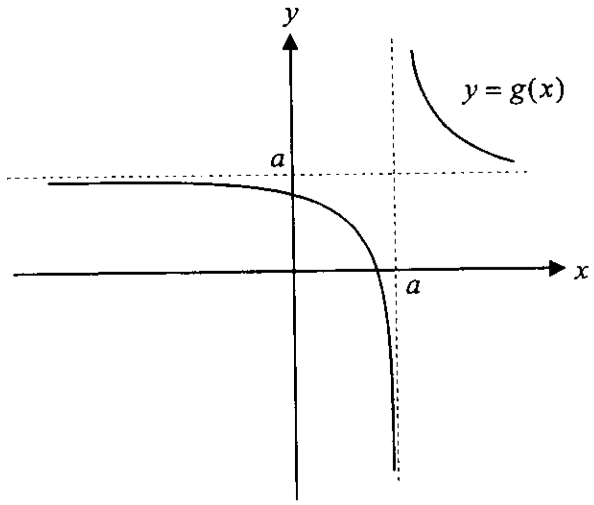
(1 mark)

ii. Since $a \geq 2$, the least value that the x-intercept can have is $x = -\dfrac{1}{2} + 2 = \dfrac{3}{2}$.

(1 mark)

(Note if $a = 3$, $x = 2\dfrac{2}{3}$ and so on. As $a \to \infty$, $x \to \infty$ and so the minimum value of x is achieved by substituting the minimum value of a which is 2.)

iii.



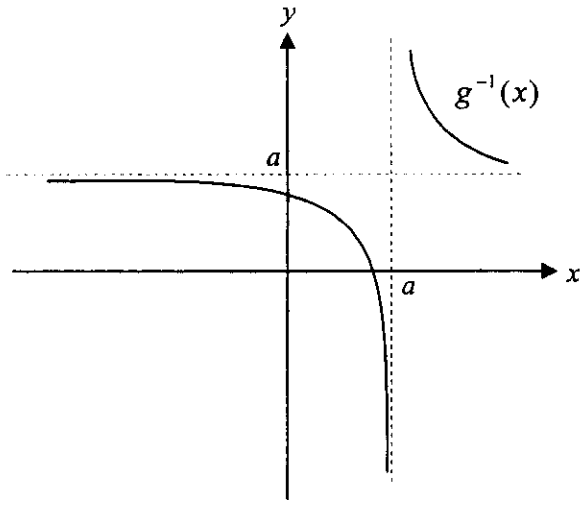
(1 mark) asymptotes
(1 mark) shape of graph including showing correct intercepts, that is, the intercepts must be positive

iv. From the graph in part **iii.**, $d_g = (-\infty, a) \cup (a, \infty)$.
Alternatively, $d_g = R \setminus \{a\}$.

(1 mark)

b. Again, from the graph in part **iii.**, we see that $g(x)$ is a one-to-one function since there is no horizontal line that can be drawn on any part of $y = g(x)$ which crosses the function more than once. **(1 mark)**

c.



(1 mark) asymptotes
(1 mark) function

Note that this is the same graph as that of $y = g(x)$ since it is a reflection of the graph of $y = g(x)$ in the line $y = x$.

d. $g^{-1}(x) = \dfrac{1}{x-a} + a, \quad a \geq 2$ from part **c.**

Alternatively, let $y = \dfrac{1}{x-a} + a$

Swap x and y. $\quad x = \dfrac{1}{y-a} + a$

$$x - a = \frac{1}{y-a}$$

$$(x-a)(y-a) = 1$$

$$y - a = \frac{1}{x-a}$$

$$y = \frac{1}{x-a} + a \text{ and so } g^{-1}(x) = \frac{1}{x-a} + a \quad a \geq 2$$ **(1 mark)**

e. **i.** $\displaystyle\int_{2a}^{3a} g(x)\,dx$

$$= \int_{2a}^{3a}\left(\frac{1}{x-a} + a\right)dx$$

$$= \left[\log_e(x-a) + ax\right]_{2a}^{3a}$$ **(1 mark)**

$$= \left\{\left(\log_e(2a) + 3a^2\right) - \left(\log_e(a) + 2a^2\right)\right\}$$

$$= \log_e(2a) - \log_e(a) + a^2$$

$$= \log_e\left(\frac{2a}{a}\right) + a^2$$

$$= \log_e 2 + a^2$$

(1 mark)

ii. Now since

$$\int_{2a}^{3a} g(x)dx = \log_e 2e^5$$

$$\log_e 2 + a^2 = \log_e 2e^5 \quad \text{(from part i.)}$$

So, $\log_e 2 + a^2 = \log_e 2 + \log_e e^5$

$$\log_e 2 + a^2 = \log_e 2 + 5\log_e e$$

$$\log_e 2 + a^2 = \log_e 2 + 5$$

(1 mark)

So, $a^2 = 5$

$a = \sqrt{5}$ since $a \geq 2$

as required **(1 mark)**

Total 14 marks

Question 3

a. d is the y-intercept.

$f(x) = 0 - 0 + 0 + 2.125$

So $d = 2.125$

(1 mark)

b. From the graph, the hose is closest to the ground over that section of $f(x)$ for which

$$f(x) = \frac{x^3}{2} - 2x^2 + 1.125x + 2.125$$

$$f'(x) = \frac{3x^2}{2} - 4x + 1.125 = 0 \quad \textbf{(1mark)}$$

$$3x^2 - 8x + 2.25 = 0$$

$$12x^2 - 32x + 9 = 0$$

$$x = \frac{32 \pm \sqrt{592}}{24}$$

$$= 2.3471 \text{ or } 0.3195$$

So $x = 0.3195$ pertains to the maximum turning point but $x = 2.3471$ pertains to the minimum turning point, which is what we are looking for.

(1 mark)

So, $f(2.3471) = 0.2127$.

So the minimum distance of the hose from the ground is 0.2 metres (correct to 1 decimal place). **(1 mark)**

c. If the rate of change of the function $f(x)$ with respect to x is positive, then the graph of the function slopes up to the right.

From part **b.** we know that the maximum turning point (where the rate of change is zero) occurs when $x = 0.3195$ and the minimum turning point (where the rate of change is also zero) occurs where $x = 2.3471$.

Between those two points the graph slopes up to the left.

So, the required answer correct to 1 decimal place where applicable, is

$$x \in [0,\ 0.3] \cup [2.3,\ 4]$$

(1 mark) **(1 mark)**

The answer $x \in [0,\ 0.3] \cup (2.3, 4]$ is also acceptable since clearly at $x = 2.3$, $\frac{dy}{dx}$ is negative. The rounding has caused the inclusion of some values which have a negative gradient.

d.

$$\text{Average rate of change} = \frac{f(4)-f(3)}{4-3} \quad \textbf{(1 mark)}$$
$$= \frac{1+\log_e 2 - 1}{1}$$
$$= \log_e 2$$

(1 mark)

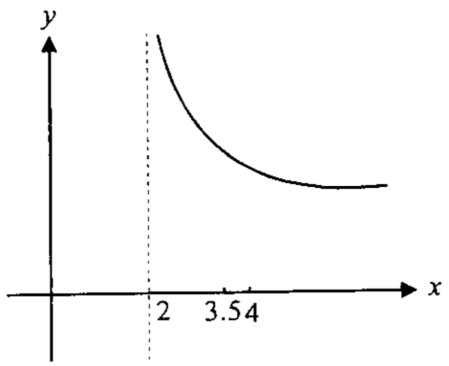
e. Over the interval $x \in [3.5, 4]$, $f(x) = 1 + \log_e(x-2)$.

The gradient or slope of the graph $y = f(x)$ and hence of the hose, is given by

$f'(x) = \frac{1}{x-2}$.

(1 mark)

The graph of $y = \frac{1}{x-2}$, $x > 2$ is shown.



(1 mark)

Over the interval $x \in [3.5, 4]$, this graph is the same as that of $y = f'(x)$.
Over the interval $x \in [3.5, 4]$, we see that the value therefore of $f'(x)$ is greatest at $x = 3 \cdot 5$. That is, as x increases in value, the value of $f'(x)$, or y, decreases in value. **(1 mark)**

So, at $x = 3.5$, $f'(3.5) = \frac{1}{1.5}$
$$= \frac{2}{3}$$

So the steepest slope of the hose over the interval $x \in [3.5, 4]$ is $\frac{2}{3}$.

(1 mark)
Total 12 marks

Question 4

a. The function $y = 2\sin\frac{\pi}{6}(x-3) + 2$ has an amplitude of 2 and is raised 2 units up. So, the value of a, the maximum height of the sheeting above the horizontal timber support is 4 cm. **(1 mark)**

b. The period of the function $y = 2\sin\frac{\pi}{6}(x-3) + 2$ is $2\pi \div \frac{\pi}{6} = 12$ **(1 mark)**

So, from Figure A, we see that the width of the corrugated plastic sheeting takes in 5 periods of the graph. So the width is $5 \times 12 = 60\text{cm}$, that is $b = 60$. **(1 mark)**

c. <u>Method 1</u>

Now, $2\sin\frac{\pi}{6}(x-3)+2=1 \qquad 6\le x\le 18$

So, $3\le x-3\le 15$

So, $\frac{3\pi}{6}\le\frac{\pi}{6}(x-3)\le\frac{15\pi}{6}$

So, $\frac{\pi}{2}\le\frac{\pi}{6}(x-3)\le\frac{5\pi}{2}$

So, to find all the relevant angles we look between $\frac{\pi}{2}$ and $\frac{5\pi}{2}$.

So, $\sin\frac{\pi}{6}(x-3)=-\frac{1}{2}$

S	A
T	C

$\frac{\pi}{6}(x-3)=\frac{7\pi}{6},\frac{11\pi}{6}$ **(1 mark)**

These 2 angles are the only angles between $\frac{\pi}{2}$ and $\frac{5\pi}{2}$ for which sine is negative and equal to $-\frac{1}{2}$.

So, $x-3=\frac{7\pi}{6}\div\frac{\pi}{6},\ \frac{11\pi}{6}\div\frac{\pi}{6}$

$=7,\ 11$

So, $x=10$ or $x=14$ **(1 mark)**

<u>Method 2</u>

We know that the period of the graph is 12 from part **b**. So, by looking at the graph in Figure A, we see that there would be two points of intersection between the graphs of $y=2\sin\frac{\pi}{6}(x-3)+2$ and $y=1$

S	A
T	C

Now, $2\sin\frac{\pi}{6}(x-3)+2=1 \qquad 6\le x\le 18$

$\sin\frac{\pi}{6}(x-3)=-\frac{1}{2}$

$\frac{\pi}{6}(x-3)=\frac{-5\pi}{6},\frac{-\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$ **(1 mark)**

$x-3=-5,\ -1,\ 7,\ 11$

$x=-2,\ 2,\ 10,\ 14$

So, $x=10$ or $x=14$ since $6\le x\le 18$. **(1 mark)**

d. The shaft of the nail enters the timber at the point where the timber makes contact with the corrugated plastic sheeting. From part **b.** we know that the period of the graph is 12 and hence the nail enters the timber 12cm from the edge. Since the shape of the plastic sheeting is symmetrical about this 12cm mark, and since the horizontal width of the cap on the nail is 4cm, the points C and D will have the x-coordinates of 10 and 14 respectively. **(1 mark)**

When $x=10$, $y=2\sin\frac{\pi}{6}(10-3)+2=1$

By symmetry, when $x=14$, $y=1$ also. So C is the point $(10,1)$ and D is the point $(14,1)$. **(1 mark)**

(Note that this result is confirmed by your answer to part **c.**)

e. From part **d.**, $m = 10$ and $n = 14$. **(1 mark)**

f. **i.** Let $y = \cos\frac{\pi}{6}(x-3)$ and let $u = \frac{\pi}{6}(x-3)$

$$= \frac{\pi x}{6} - \frac{\pi}{2}$$

So, $y = \cos u$ $\qquad \frac{du}{dx} = \frac{\pi}{6}$

$$\frac{dy}{du} = -\sin u$$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (Chain rule)

$$= -\sin u \cdot \frac{\pi}{6}$$

$$= \frac{-\pi}{6} \sin\frac{\pi}{6}(x-3)$$

(1 mark)

ii. Now, $\int 2\sin\frac{\pi}{6}(x-3)dx$

$$= 2\int \sin\frac{\pi}{6}(x-3)dx$$

$$= \frac{-6}{\pi} \times 2\int \frac{-\pi}{6} \sin\frac{\pi}{6}(x-3)dx$$

$$= \frac{-12}{\pi} \int \frac{-\pi}{6} \sin\frac{\pi}{6}(x-3)dx$$ **(1 mark)**

Hence,

$$= \frac{-12}{\pi} \cos\frac{\pi}{6}(x-3) + c$$ from part **i.**

(1 mark)

g. Area required $= \int_{10}^{14} \left\{ \left(\frac{-(x-12)^2}{4} + 2 \right) - \left(2\sin\frac{\pi}{6}(x-3) + 2 \right) \right\} dx$

(1 mark) for terminals
(1 mark) for function

$$= \int_{10}^{14} \left\{ \left(\frac{-(x^2 - 24x + 144)}{4} + 2 \right) - 2\sin\frac{\pi}{6}(x-3) - 2 \right\} dx$$

$$= \int_{10}^{14} \left(\frac{-x^2}{4} + 6x - 36 - 2\sin\frac{\pi}{6}(x-3) \right) dx$$

$$= \left[\frac{-x^3}{12} + \frac{6x^2}{2} - 36x + \frac{12}{\pi}\cos\frac{\pi}{6}(x-3) \right]_{10}^{14}$$ from part **f.ii.**

(1 mark) for cubic function
(1 mark) for trigonometric function

$= 5.3\text{cm}^2$ **(1 mark)**

Total 16 marks