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VCE Mathematical Methods Units 3 & 4

Written Examination 2: Analysis task

Suggested Solutions

Question 1

a. $a = 2,\ b = 0,\ c = 4$ [A] [A] [A]

b. Note that the period = 12

$$\therefore \frac{2\pi}{n} = 12$$

$$\therefore n = \frac{2\pi}{12} = \frac{\pi}{6}$$ [A]

c. When $t = 5$, $D(t) = 2\sin\frac{5\pi}{6} + 4$ [M]

$$= 2 \times \frac{1}{2} + 4$$

$$= 5$$ [A]

d. The yacht must use a depth of at least 3 m, so when $D = 3$ we have

$$2\sin\frac{\pi}{6}t + 4 = 3$$ [M]

$$2\sin\frac{\pi}{6}t = -1$$

$$\sin\frac{\pi}{6}t = -\frac{1}{2}$$ [A]

$$\therefore \frac{\pi}{6}t = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$

$$\therefore \frac{\pi}{6}t = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\therefore t = 7, 11$$ [A][A]

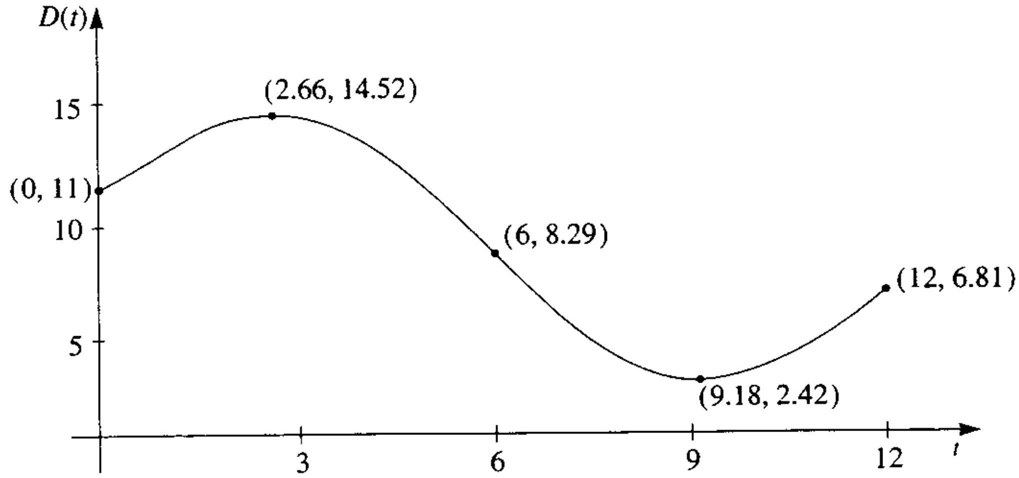
Hence the yacht is unable to sail for 4 hours between 7 a.m. and 11 a.m.

e.

t	$D(t)$		
0	11.00	$6e^{0} + 5\sin 0 + 5$	[A]
6	8.29	$6e^{-0.6} + 5\sin\pi + 5$	[A]
12	6.81	$6e^{-1.2} + 5\sin 2\pi + 5$	[A]

f. Using a graphic calculator or by equating the derivative to zero (not advised):
First maximum = (2.66, 14.52) [A]
First minimum = (9.18, 2.42) [A]

g.



The graph should indicate the points
(0, 11), (2.66, 14.52), (6, 8.29), (9.18, 2.42) and (12, 6.81) [A] [A]

Question 2

a. $p = 1000$ [A]

b. Substitute the point (1300, 500) to obtain $500 = Q1300(300)^2$ [M]

$$\therefore Q = \frac{500}{1300(300)^2}$$

$$= 4.3 \times 10^{-6}$$ [A]

c. If we are assuming x is measured in kilometres, gradient $= \frac{dy}{dx} = 4.3 \times 10^{-6}(3x^2 - 4x + 1)$ [A]

(using either product rule or differentiating the expanded form) [M]

(If assuming x is in metres, gradient $= 4.3 \times 10^{-6}(3x^2 - 4000x + 1000000)$.)

d. At B, $\frac{dy}{dx} = 0$ (using x measured in kilometres)

$\therefore 3x^2 - 4x + 1 = 0$ (since $Q \neq 0$) [M]

$(3x - 1)(x - 1) = 0$

$\therefore x = \frac{1}{3}$ and 1 kilometre

or 333 m and 1000 m [A]

Hence the coordinates of B (using graphic calculator) are (333, 637) [A]

e. At C, $x = 700$

Gradient $= \frac{dy}{dx} = 4.3 \times 10^{-6}(3(700) - 1000)(700 - 1000)$

(by substituting into the factorized form) [M]

$$\frac{dy}{dx} = -1.419$$ [A]

f. For greatest slope, we can differentiate the gradient function and equate to zero, obtaining

$\frac{d^2y}{dx^2} = 6x - 4 = 0$ (if x is in kilometres). [M]

$\therefore x = \frac{2}{3}$ kilometres or 667 metres [A]

(use of graphic calculator and tracing with derivative function turned on could also be used)

Question 3

a. As $\Sigma \Pr(X = x) = 1$, $m = \Pr(X = 3) = 0.15$ [A]

b. $E(X) = \Sigma x \times p(x)$

$= (1 \times 0.4) + (2 \times 0.25) + (3 \times 0.15) + (4 \times 0.1) + (5 \times 0.1)$ [M]

$= 2.25$ [A]

c. $\text{Var}(X) = E(X^2) + (E(X))^2$

$E(X^2) = (1 \times 0.4) + (4 \times 0.25) + (9 \times 0.15) + (16 \times 0.1) + (25 \times 0.1)$ [M]

$= 0.40 + 1.0 + 1.35 + 1.6 + 2.5$

$= 6.85$ [A]

so $\text{Var}(X) = 6.85 - 2.25^2$

$= 1.7875$

≈ 1.788 [A]

d. $\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{1.7875} = 1.337$ [A]

e. As $Y = 2X - 1$, $E(Y) = E(2X - 1)$

$= 2E(X) - 1$

$= 2 \times 2.25 - 1$

$= 3.50$ [A]

i. $\text{Var}(Y) = \text{Var}(2X - 1)$

$= 4\text{Var}(X)$

$= 4 \times 1.7875$

$= 7.150$ [A]

f.

Z	**0**	**1**	**2**	**3**
$\Pr(Z = z)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$

i. $c = \Pr(Z = 1) = {}^3C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = 3 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ [A]

$d = \Pr(Z = 2) = {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 = 3 \times \frac{1}{9} \times \frac{2}{3} = \frac{2}{9}$ [A]

ii. $\text{SD}(Z) = \sqrt{\text{Var}(Z)} = \sqrt{npq} = \sqrt{3 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$ [A]

g. Letting B = "number of boys selected"

then $\Pr(B = 2) = \dfrac{({}^4C_2)({}^4C_1)}{({}^8C_3)} = 0.4286$ [A]

Question 4

a. $x = -a$ is the vertical asymptote.

This is determined by equating the denominator of $p(x)$ to zero, i.e. $x = -1$

$\therefore a = 1$ [A]

$y = b$ is the y intercept.

Substitute $x = 0$ into $p(x)$: $y = \dfrac{-e^{x-1}}{x+1}$, where $x = 0$

$\therefore y = \dfrac{-e^{-1}}{1} = \dfrac{-1}{e}$

$\therefore b = \dfrac{-1}{e}$ [A]

b. Using the quotient rule: [M]

$p'(x) = \dfrac{(x+1)(-e^{x-1}) - (-e^{x-1}) \times 1}{(x+1)^2}$

$= \dfrac{-e^{x-1}[(x+1) - 1]}{(x+1)^2}$ [M]

$= \dfrac{-xe^{x-1}}{(x+1)^2}$ [A]

c. $p'(x) = \dfrac{-xe^{x-1}}{(x+1)^2} = 0$ at the stationary point [M]

$-xe^{x-1} = 0$ (since $(x+1)^2 \neq 0$)

$\therefore x = 0$ (since $e^{x-1} \neq 0$)

At $x = 0$, $y = b$ (from **a.**)

So a stationary point occurs at $(0, b)$ [A]

d. Either by use of the second derivative, or by testing the gradient of the curve on either side of the stationary point, or by substituting into $p(x)$. For example:

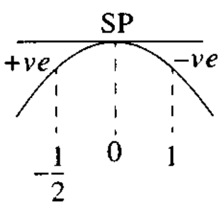
At $x = -\frac{1}{2}$, $p'(x) = \dfrac{\frac{1}{2}e^{(-1/2)-1}}{\left(-\frac{1}{2}+1\right)^2}$

$= \dfrac{\frac{1}{2}e^{-3/2}}{\frac{1}{4}}$

≈ 0.45 (positive)

At $x = 1$, $p'(1) = \dfrac{-1e^0}{(1+1)^2}$

$= -\frac{1}{4}$ (negative)



[M]

So the stationary point is a maximum. [A]

e. Required area $= \int_0^1 \left[p'(x) - \left(4x - \frac{17}{4}\right)\right] dx$ [A]

$= \int_0^1 \left(\frac{-xe^{x-1}}{(x+1)^2} - 4x + \frac{17}{4}\right) dx$ [A]

$= \left[\frac{-e^{x-1}}{x+1} - 2x^2 + \frac{17}{4}x\right]_0^1$ [M]

$= \left(-\frac{e^0}{2} - 2 + \frac{17}{4}\right) - \left(-\frac{e^{-1}}{1} - 0 + 0\right)$ [M]

$= -\frac{1}{2} - 2 + \frac{17}{4} + \frac{1}{e}$

$= \frac{7}{4} + \frac{1}{e}$ square units [A]