Trial Examination 2001

VCE Mathematical Methods Units 3 & 4

Written Examination 1: Facts, skills and applications task

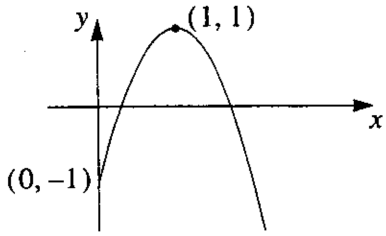
Suggested Solutions

PART I

Question 1

We can make use of the turning point form to sketch the graph of $y = 1 - 2(x-1)^2$.

That is, $y = 1 - 2(x-1)^2 = -2(x-1)^2 + 1$



From the graph, the range is seen to be $(-\infty, 1]$.

Answer D

Question 2

Turning points at $x = -2$ and $x = 2$ means that we have factors of the form $(x+2)^2$ and $(x-2)^2$.

Therefore, the equation takes on the form $y = a(x+2)^2(x-2)^2, a \in R$

Next, when $x = 0$, y is positive. This means that a is also positive.

We can then assume that $y = (x+2)^2(x-2)^2$

$$y = [(x+2)(x-2)]^2$$

$$y = (x^2 - 4)^2$$

Answer E

Question 3

Using Pascal's triangle, we have $\left(2x - \frac{3}{x}\right)^4 = (2x)^4 - 4(2x)^3\left(\frac{3}{x}\right) + 6(2x)^2\left(\frac{3}{x}\right)^2 - 4(2x)\left(\frac{3}{x}\right)^3 - \left(\frac{3}{x}\right)^4$

By observation, the term independent of x is the 3rd term.

i.e. $6(2x)^2\left(\frac{3}{x}\right)^2 = 6 \times 4 \times 9 = 216$

Answer A

Question 4

First: $(0, 0) \rightarrow (1, 0)$ therefore translation of 1 unit to the right, i.e. $f(x) \rightarrow f(x-1)$, so that the point $(1, 1)$ would now be (2, 1).

However we have the point (2, 8), indicating a dilation parallel to y-axis.

Therefore $f(x-1) \rightarrow 8f(x-1)$, i.e. $f(x) \rightarrow f(x-1) \rightarrow 8f(x-1)$

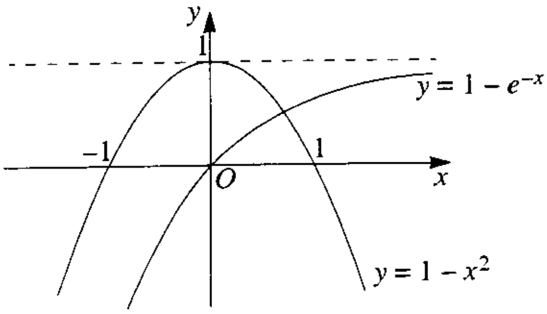
Answer A

Question 5

Reflect the graph about the line $y = x$. One quick check is to select a few coordinates, interchange the x and y values and plot them. The graph of the inverse should pass through these new points.

Answer E

Question 6



From graph, the only solution is in $[0, 1]$.

Answer B

Question 7

$$\log_{10}x = \log_{10}(by - a) - \log_{10}a$$

$$\Leftrightarrow \log_{10}x = \log_{10}\left(\frac{by - a}{a}\right)$$

$$\Leftrightarrow x = \frac{by - a}{a}$$

$$\Leftrightarrow ax = by - a$$

$$\Leftrightarrow by = a(x + 1)$$

$$\Leftrightarrow y = \frac{a}{b}(x + 1)$$

Answer B

Question 8

First we investigate the dilation factor (parallel to y-axis):

$\frac{a}{b} \to \frac{1}{2}ab$, i.e. we need to multiply $\frac{a}{b}$ by $\frac{1}{2}b^2$ to get $\frac{1}{2}ab$

Therefore, $f(x) \to \frac{1}{2}b^2 f(x)$

Next, dilation parallel to x-axis: $2\pi \to \pi$

Therefore, $f(x) \to f(2x)$

$\therefore f(x) \to \frac{1}{2}b^2 f(2x)$

Answer E

Question 9

$f(x) = -a + 5a\sin(c\pi x)$

Min is $-5a - a = -6a$

Max is $5a - a = 4a$

Period is $\frac{2\pi}{c\pi} = \frac{2}{c}$

Answer B

Question 10

$$\text{Slope of } PQ = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{x - x - h}{h(x)(x+h)}$$

$$= -\frac{1}{x(x+h)}$$

Answer C

Question 11

By the chain rule, the derivative is $-2\sin 2x \times e^{\cos 2x}$

Answer C

Question 12

Using the quotient rule, $$\frac{dy}{dx} = \frac{e^{2x}\frac{1}{2}(2x-1)^{-1/2} \times 2 - \sqrt{2x-1} \times 2e^{2x}}{(e^{2x})^2}$$

$$= \frac{1}{e^{2x}\sqrt{2x-1}} - \frac{2\sqrt{2x-1}}{e^{2x}}$$

$$= \frac{1 - 2(2x-1)}{e^{2x}\sqrt{2x-1}}$$

$$= \frac{3 - 4x}{e^{2x}\sqrt{2x-1}}$$

Answer A

Question 13

$V'(t) = \frac{3}{20}\cos\frac{t}{10}$

At $t = 5$, $V'(5) = \frac{3}{20}\cos\frac{1}{2} = 0.13$

Answer A

Question 14

The approximate change in y, δy, is given by $\delta y = \frac{dy}{dx}\delta x$.

$\frac{dy}{dx} = 3x^2 + 1$

At $x = 2$, $\frac{dy}{dx} = 13$

So $\delta y = 13 \times 0.01$

$= 0.13$

Answer D

Question 15

We can set up a table of values:

x	1	2	3	4
y	2	1	$\frac{2}{3}$	$\frac{1}{2}$

$$\text{Required area } = (2\times 1) + (1\times 1) + \left(\frac{2}{3}\times 1\right) + \left(\frac{1}{2}\times 1\right)$$

$$= 4\frac{1}{6} \text{ sq. units}$$

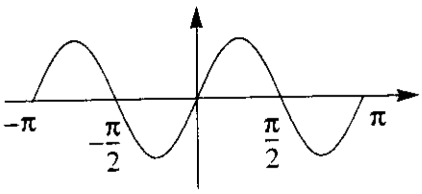
Answer A

Question 16

Remember the areas below the x axis are negative integrals so we must subtract this area.

Answer C

Question 17



$$A = 4\int_0^{\frac{\pi}{2}} 3\sin 2x\, dx$$

Answer D

Question 18

Four or more customers were served on 16 days. Therefore, proportion is $\frac{16}{25} = 0.64$.

Answer A

Question 19

We can generate the probability distribution function (table) for X:

x	0	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{2}{25}$	0	$\frac{4}{25}$	$\frac{3}{25}$	$\frac{2}{25}$	$\frac{6}{25}$	$\frac{8}{25}$

$$E(X) = \frac{1}{25}(2\times 4 + 3\times 3 + 4\times 2 + 5\times 6 + 6\times 8)$$

$$= \frac{1}{25}(8 + 9 + 8 + 30 + 48)$$

$$= 4.12$$

Answer C

Question 20

$n = 12$, $E(x) = np = 7.2$, therefore

$$p = \frac{7.2}{12} = 0.6$$

$$\begin{aligned}\text{Var}(x) &= npq \\ &= 12 \times 0.6 \times 0.4 \\ &= 2.88\end{aligned}$$

Answer A

Question 21

The fact that the guessing process is repeated 27 times implies a binomial process.

Here $n = 27$, $p = \frac{1}{5}$, $q = \frac{4}{5}$, $x = 20$.

So if x = number of correct guesses,

$$\Pr(x = 20) = {}^{27}C_{20}\left(\frac{1}{5}\right)^{20}\left(\frac{4}{5}\right)^{7}$$

Answer E

Question 22

$$\begin{aligned}\frac{{}^{8}C_{1}\,{}^{12}C_{3}}{{}^{20}C_{4}} + \frac{{}^{8}C_{0}\,{}^{12}C_{4}}{{}^{20}C_{4}} &= 0.3633 + 0.1022 \\ &= 0.4655 \\ &= 0.47 \text{ correct to 2 decimal places}\end{aligned}$$

Answer B

Question 23

$$\begin{aligned}\text{Var}(X) &= n\frac{D}{N}\left(1 - \frac{D}{N}\right)\left(\frac{N-n}{N-1}\right) \\ &= 4 \times \frac{12}{20} \times \frac{8}{20} \times \frac{16}{19} \\ &= \frac{384}{475}\end{aligned}$$

$$\begin{aligned}\sigma(X) &= \sqrt{\frac{384}{475}} \\ &\approx 0.90\end{aligned}$$

Answer C

Question 24

With $Y = 2Z + 1$, $E(Y) = E(2Z + 1) = 2E(Z) + 1 = 1$ (since $E(Z) = 0$).
Therefore the graph is translated 1 unit to the right.

Next, $\text{Var}(Y) = \text{Var}(2Z + 1)$
$= \text{Var}(2Z)$
$= 4\,\text{Var}(Z)$

Therefore distribution of Y has a much larger spread.

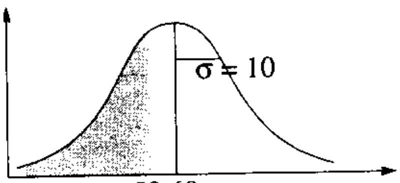
Answer C

Question 25

The 95% confidence interval is given by $(\mu - 2\sigma, \mu + 2\sigma)$,
i.e. $(132 - 2\sqrt{9}, 132 + 2\sqrt{9})$ or $(126, 138)$.

Answer B

Question 26



Let X = time taken to answer.

$$\Pr(X < 50) = \Pr\left(Z < \frac{50 - 60}{10}\right)$$
$$= \Pr(Z < -1)$$

Answer A

Question 27

Using X = time taken to answer,

$$\Pr(X < 50 | X < 60) = \frac{\Pr(X < 50 \cap X < 60)}{\Pr(X < 60)}$$
$$= \frac{\Pr(X < 50)}{\Pr(X < 60)}$$
$$= \frac{0.1587}{0.5}$$
$$= 0.3174$$

Answer D

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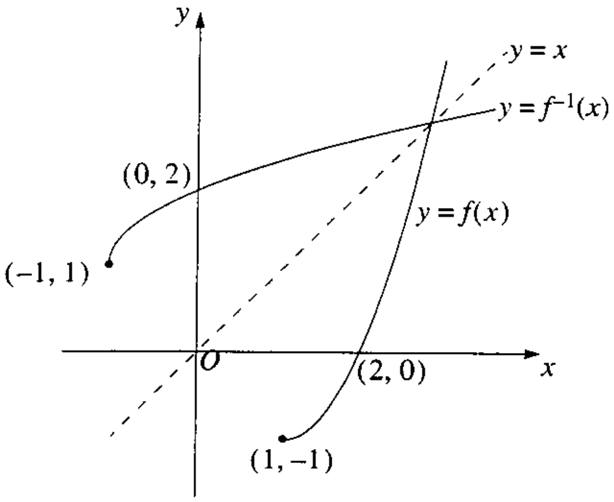
PART II

Question 1

a.



[A] [A]

b. Original function: $y = x^2 - 2x$

$= (x-1)^2 - 1$

Inverse function: $x = (y-1)^2 - 1$

$x + 1 = (y-1)^2$ [M]

$\sqrt{x+1} = y - 1$

(only the positive square root is needed due to restricted domain of f)

$\therefore f^{-1}(x) = \sqrt{x+1} + 1$ [A]

c. See graph in **a.** [A] [A]

Question 2

a. $\dfrac{d(\sin 2x^2)}{dx} = 4x\cos(2x^2)$

Use the chain rule: Let $u = 2x^2$, so that $\dfrac{du}{dx} = 4x$.

Then $y = \sin u$, $\dfrac{dy}{du} = \cos u$. [M]

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= \cos u \times 4x$

$= 4x\cos(2x^2)$ [A]

b. As $\dfrac{d(\sin(2x^2))}{dx} = 4x\cos(2x^2)$,

$\int 4x\cos(2x^2)\,dx = \sin(2x^2) + c$ [M]

$\therefore \int x\cos(2x^2)\,dx = \frac{1}{4}\sin(2x^2) + c$ [A]

Question 3

a. When $x = e$, $y = \log_e e^2$

$= 2\log_e e$

$= 2$ [A]

b. Gradient of tangent $= \dfrac{dy}{dx}$

$= \dfrac{2x}{x^2}$

$= \dfrac{2}{x}$

When $x = e$, gradient $= \dfrac{2}{e}$ [A]

c. Gradient of normal $= -\dfrac{e}{2}$ [A]

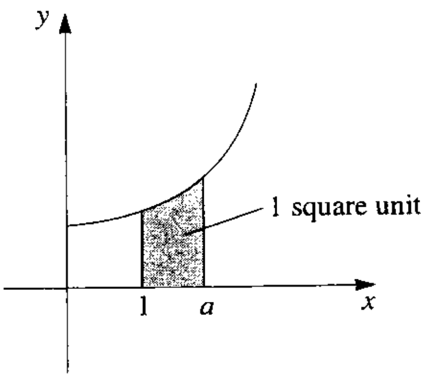
Equation of normal is $y = \dfrac{-e}{2}x + c$

Substitute $(e, 2)$: $2 = \dfrac{-e^2}{2} + c$ [M]

$\therefore c = 2 + \dfrac{e^2}{2}$

Hence equation required is $y = \dfrac{-e^2}{2}x + \dfrac{e^2}{2} + 2$ [A]

Question 4



Area $= \displaystyle\int_1^a x^2 + a \; dx$

$\therefore 1 = \left[\dfrac{x^3}{3} + ax\right]_1^a$ [M]

$1 = \left(\dfrac{a^3}{3} + a^2\right) - \left(\dfrac{1}{3} + a\right)$ [A]

$\therefore 3 = a^3 + 3a^2 - 1 - 3a$

or $a^3 + 3a^2 - 3a - 4 = 0$ [A]

Using a graphic calculator, $a = 1.361$ [A]

Question 5

a. Let X = the number of black balls. [M]

$$\Pr(X=3) = \frac{({}^{3}C_{3})({}^{7}C_{2})}{{}^{10}C_{5}}$$

$$= 0.083$$ [A]

b. The first three balls drawn are white. This leaves 3 black and 4 white balls, from which 2 are to be drawn. If X = the number of black balls selected, we require

$$\Pr(X \geq 1) = 1 - \Pr(X=0)$$ [M]

$$= 1 - \frac{({}^{3}C_{0})({}^{4}C_{2})}{({}^{7}C_{2})}$$

$$= 1 - \frac{6}{21}$$

$$= \frac{15}{21}$$ [A]

Or alternatively, $\Pr(X \geq 1) = \Pr(X=1) + \Pr(X=2)$

$$= \frac{({}^{3}C_{1})({}^{4}C_{1})}{({}^{7}C_{2})} + \frac{({}^{3}C_{2})({}^{4}C_{0})}{({}^{7}C_{2})}$$

$$= \frac{12}{21} + \frac{3}{21}$$

$$= \frac{15}{21} \text{ (or 0.714)}$$