THE HEFFERNAN GROUP

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MATHS METHODS 3 & 4 TRIAL EXAMINATION 2 2001 SOLUTIONS

Question 1

a. Since the graph passes through $(1, 2.5)$, we have $2.5 = \frac{k}{(1+1)^2}$

$$2.5 = \frac{k}{4}$$

$k = 10$ as required **(1 mark)**

b. The *x* intercept occurs when $y = 0$

So, $$y = \frac{10x}{(x^2+1)^2}$$

becomes $$0 = \frac{10x}{(x^2+1)^2}$$

Now, $(x^2+1)^2 \neq 0$ or else *f* is undefined.

So, $10x = 0$

$x = 0$ as required **(1 mark)**

c. i. $$\frac{d}{dx}\left(\frac{10x}{(x^2+1)^2}\right) = \frac{(x^2+1)^2 \times 10 - 10x \times 2(x^2+1)^1 \times 2x}{(x^2+1)^4}$$ **(1 mark)**

$$= \frac{10(x^2+1)^2 - 40x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{10(x^2+1)\{(x^2+1) - 4x^2\}}{(x^2+1)^4}$$

$$= \frac{10(x^2+1-4x^2)}{(x^2+1)^3}$$

$$= \frac{10(1-3x^2)}{(x^2+1)^3}$$ as required **(1 mark)**

ii. The turning point(s) of the graph of $y = f(x)$ occur when $f'(x) = 0$

We have $\frac{10(1-3x^2)}{(x^2+1)^3} = 0$ **(1 mark)**

Now, $(x^2+1) \neq 0$ so $10(1-3x^2) = 0$

$$10(1-\sqrt{3}x)(1+\sqrt{3}x) = 0$$

$$x = \pm\frac{1}{\sqrt{3}}$$ **(1 mark)**

Now, $f(\frac{1}{\sqrt{3}}) = \frac{10}{\sqrt{3}} \div (\frac{1}{3}+1)^2$

$= \frac{10}{\sqrt{3}} \times \frac{9}{16}$

$= \frac{45}{8\sqrt{3}}$

$= \frac{45\sqrt{3}}{8 \times 3}$

$= \frac{15\sqrt{3}}{8}$

Also, $f(-\frac{1}{\sqrt{3}}) = \frac{-10}{\sqrt{3}} \div (\frac{1}{3}+1)^2$

$= \frac{-15\sqrt{3}}{8}$

So the turning points are $(\frac{1}{\sqrt{3}}, \frac{15\sqrt{3}}{8})$ and $(\frac{-1}{\sqrt{3}}, \frac{-15\sqrt{3}}{8})$ **(1 mark)**

iii. Looking at the graph and using the turning points found in part **ii**., we have,

$r_f = \left[\frac{-15\sqrt{3}}{8}, \frac{15\sqrt{3}}{8}\right]$ **(1 mark)**

d. **i.** $\frac{d}{dx}\left(-5(x^2+1)^{-1}\right) = 5(x^2+1)^{-2} \times 2x$

$= \frac{10x}{(x^2+1)^2}$ **(1 mark)**

ii. Area required $= \int_0^2 \frac{10x}{(x^2+1)^2}$ **(1 mark)**

From part **d. i.** we have, $\frac{d}{dx}\left(\frac{-5}{x^2+1}\right) = \frac{10x}{(x^2+1)^2}$

So, since area required $= \int_0^2 \frac{10x}{(x^2+1)^2}\,dx$

we have $= \left[\frac{-5}{x^2+1}\right]_0^2$ **(1 mark)**

$= (\frac{-5}{5}) - (\frac{-5}{1}) = 4$ So area required is 4 square units **(1 mark)**

Total 12 marks

Question 2

a. $\frac{12}{30} = \frac{2}{5}$ **(1 mark)**

b. Once the track is played, it cannot be repeated and so there is no "replacement" of tracks and so we have a hypergeometric distribution.

So, $\Pr(X = 2) = \frac{{}^{12}C_2 \times {}^{18}C_3}{{}^{30}C_5}$ **(1 mark)**

$= \frac{66 \times 816}{142\,506} = 0.38$ to 2 decimal places **(1 mark)**

c. $\Pr(X \geq 1) = 1 - \Pr(X = 0)$ **(1 mark)**

$= 1 - \left(\frac{{}^{18}C_0 \times {}^{12}C_5}{{}^{30}C_5}\right)$

$= 1 - \frac{1 \times 792}{142\,506} = 0.9944$ to 4 decimal places **(1 mark)**

d. The expected number of Dean Martin tracks $= n\frac{D}{N}$

$= 5 \times \frac{12}{30} = 2$ **(1 mark)**

e. Because the tracks can be chosen again and again, they are effectively being replaced. So, we have a binomial distribution.

So, $\Pr(X = 4) = {}^{8}C_4\left(\frac{12}{30}\right)^4\left(\frac{18}{30}\right)^4 = 0.2322$ **(1 mark)**

f. The probability of 1 particular track being played is $\frac{1}{12+18} = \frac{1}{30}$ **(1 mark)**

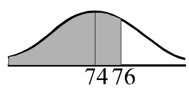
The probability of it being played twice when 8 tracks are chosen is given by

$\Pr(X = 2) = {}^{8}C_2\left(\frac{1}{30}\right)^2\left(\frac{29}{30}\right)^6$

$= 0.0254$ to 4 decimal places **(1 mark)**

g. i. $\Pr(X < 76) = \Pr(Z < 0.5)$ Now, $Z = \frac{X - \mu}{\sigma}$

$= 0.6915$ **(1 mark)** $= \frac{76 - 74}{4} = 0.5$

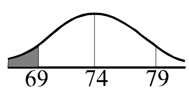


ii. $\Pr(X < 69) = \Pr(Z < -1.25)$ Now, $Z = \frac{X - \mu}{\sigma}$

$= 1 - \Pr(Z < 1.25)$ **(1 mark)** $= \frac{69 - 74}{4}$

$= 1 - 0.8944$ $= -1.25$

$= 0.1056$ **(1 mark)**



h. $\Pr(\text{at least 1 CD had to be replaced}) = 1 - \Pr(\text{neither CD had to be replaced})$ **(1 mark)**

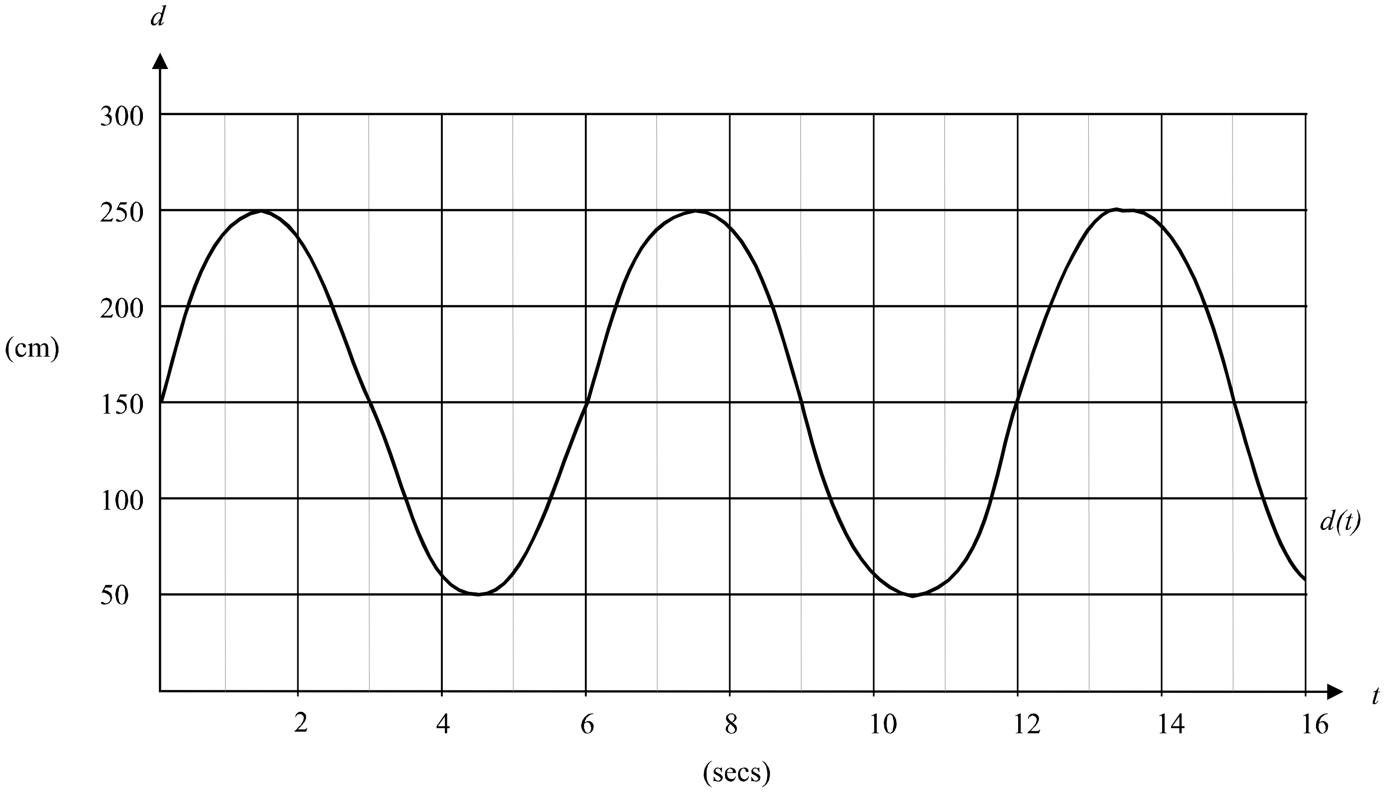
$= 1 - (0.3085)(0.8944)$ **(1 mark)**

$= 0.7241$ **(1 mark)**

Total 15 marks

Question 3

a.



(2 marks)

b. At $t = 0$, we have $d(0) = 100\sin 0 + 150$

$$= 150$$

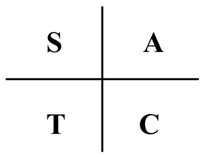
The bottom of the rope is 150cm from the ground initially. **(1 mark)**

c. From the graph, we see that the maximum distance of the bottom of the rope from the floor is 250 cm. **(1 mark)**

d. i. When $d = 100$, we have $100\sin\frac{\pi t}{3} + 150 = 100$

$$100\sin\frac{\pi t}{3} = -50$$ **(1 mark)**

$$\sin\frac{\pi t}{3} = -0.5$$



$$\frac{\pi t}{3} = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}\ldots$$

$$t = \frac{7}{2}, \frac{11}{2}, \frac{19}{2}, \frac{23}{2}\ldots$$

So the times when Georges bell first rings during the first 6 seconds are $t = 3\frac{1}{2}$ and $t = 5\frac{1}{2}$.

(1 mark)

ii. Six seconds represents 1 period of the function. From part **d. i.**, there are 2 rings in the period of 6 seconds so over 60 seconds there will be $2 \times 10 = 20$ rings **(1 mark)**

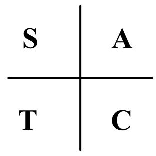
e. We want to solve $\quad d(t) = d_T(t)$

So, $\quad 100\sin\frac{\pi t}{3} + 150 = 100\sqrt{3}\cos\frac{\pi t}{3} + 150$

$$100\sin\frac{\pi t}{3} = 100\sqrt{3}\cos\frac{\pi t}{3}$$

$$\frac{\sin\frac{\pi t}{3}}{\cos\frac{\pi t}{3}} = \frac{100\sqrt{3}}{100}$$

$$\tan\frac{\pi t}{3} = \sqrt{3}$$ **(1 mark)**



$$\frac{\pi t}{3} = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3} \ldots$$

$$t = 1, 4, 7, 10, 13 \ldots$$

In the first 12 seconds, the bottom of the ropes would be the same distance from the ground at $t = 1, 4, 7$ and 10 seconds. **(1 marks)**

f. i. Use a graphics calculator to sketch the graph of $y = D(t)$ over the domain $t \in [0, 20]$.
Adjust your WINDOW to, say, Xmin = 0, Xmax = 20, Xscl = 1, Ymin = 100, Ymax = 350, Yscl = 10, so that you see the whole picture.
There are 4 'local' maxima which appear over this domain. Use the Max function on your graphics calculator to find which one of these 4 is the maximum over the domain. The maximum over the domain is located at $(7.5, 343.9)$. So the distance between the bottom of the two ropes is greatest when $t = 7.5$ secs (to 1 decimal place) **(2 marks)**

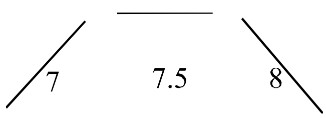
ii. Now, $\quad D(t) = 100\sin\frac{\pi t}{3} - t^2 + 20t + 150$

so, $\quad D'(t) = \frac{100\pi}{3}\cos\frac{\pi t}{3} - 2t + 20$ **(1 mark)**

At $t = 7$, $\quad D'(t) = \frac{100\pi}{3}\cos\frac{7\pi}{3} - 14 + 20 = 58.36$ (to 2 decimal places) which is a positive number.

At $t = 8$, $\quad D'(t) = \frac{100\pi}{3}\cos\frac{8\pi}{3} - 16 + 20 = -48.36$ (to 2 decimal places) which is a negative number.

So, from the diagram we see that the turning point which occurs when $t = 7.5$ (to 1 decimal place) must be a maximum.



(1 mark)

Total 13 marks

Question 4

a. The y-intercept occurs when $x = 0$.

We have $y = 2e^0 - e^0 + 10$

$= 2 - 1 + 10$

$= 11$

The y-intercept is 11. **(1 mark)**

b. The x-intercept occurs when $y = 0$.

We have $2e^{\frac{x}{2}} - e^{\frac{3x}{2}} + 10 = 0$

Use a graphics calculator to solve this. We obtain $x = 1.8$ (to 1 decimal place) **(1 mark)**

c. i. We have $g(x) = 2e^{\frac{x}{2}} - e^{\frac{3x}{2}} + 10$

So, $g'(x) = e^{\frac{x}{2}} - \frac{3}{2}e^{\frac{3x}{2}}$ **(1 mark)**

When $g'(x) = 0$, we have

$e^{\frac{x}{2}} - \frac{3}{2}e^{\frac{3x}{2}} = 0$ **(1 mark)**

$e^{\frac{x}{2}} = \frac{3}{2}e^{\frac{3x}{2}}$

$\frac{2}{3} = \frac{e^{\frac{3x}{2}}}{e^{\frac{x}{2}}}$

$e^x = \frac{2}{3}$

So, $x = \log_e \frac{2}{3}$ **(1 mark)**

ii. To find the y-coordinate of the turning point, substitute $x = \log_e \frac{2}{3}$ into $g(x)$.

So, $g(x) = 2e^{\frac{x}{2}} - e^{\frac{3x}{2}} + 10$

$= 2e^{\frac{1}{2}\log_e \frac{2}{3}} - e^{\frac{3}{2}\log_e \frac{2}{3}} + 10$

$= 2e^{\log_e (\frac{2}{3})^{\frac{1}{2}}} - e^{\log_e (\frac{2}{3})^{\frac{3}{2}}} + 10$

$= 2 \times (\frac{2}{3})^{\frac{1}{2}} - (\frac{2}{3})^{\frac{3}{2}} + 10$ **(1 mark)**

$= 2\sqrt{\frac{2}{3}} - \sqrt{(\frac{2}{3})^3} + 10$

$= 2\sqrt{\frac{2}{3}} - \sqrt{\frac{8}{27}} + 10$

So, $m = \frac{2}{3}$ and $n = \frac{8}{27}$ **(1 mark)**

d. The gradient of the function g at the point $x = 0$ is given by $g'(0)$.

From part **c. i.** We have $g'(x) = e^{\frac{x}{2}} - \frac{3}{2}e^{\frac{3x}{2}}$

So, $$g'(0) = e^0 - \frac{3}{2}e^0$$

$$= 1 - \frac{3}{2} = -\frac{1}{2}$$ **(1 mark)**

The equation of the tangent at the point $(0,11)$ is given by

$$y - y_1 = m(x - x_1)$$

$$y - 11 = \frac{-1}{2}(x - 0)$$

$$y = \frac{-x}{2} + 11$$ as required **(1 mark)**

e. The pole is located at $x = -2$ and the equation of the rope is $y = \frac{-x}{2} + 11$.

They intersect when $y = \frac{-(-2)}{2} + 11 = 12$ **(1 mark)**

So, they intersect at the point $(-2,12)$. Now, whilst the y-coordinate is 12, this is not the distance from the ground to the point where the rope is attached since the pole doesn't stand on ground at sea level. We need to find the height of the ground at $x = -2$

Now, $$g(x) = 2e^{\frac{x}{2}} - e^{\frac{3x}{2}} + 10$$

At $x = -2$, $g(x) = 2e^{-1} - e^{-3} + 10 = 10.6860$ (to 4 places) **(1 mark)**

So the rope is attached at $12 - 10.6860 = 1.31$ metres (to 2 decimal places) above the bottom of the pole. **(1 mark)**

f. Area required $= \int_{-2}^{0} \left\{ (\frac{-x}{2} + 11) - g(x) \right\} dx$ **(1 mark)**

$$= \int_{-2}^{0} \left\{ \frac{-x}{2} + 11 - (2e^{\frac{x}{2}} - e^{\frac{3x}{2}} + 10) \right\} dx$$

$$= \int_{-2}^{0} (\frac{-x}{2} + 11 - 2e^{\frac{x}{2}} + e^{\frac{3x}{2}} - 10) dx$$

$$= \int_{-2}^{0} (\frac{-x}{2} - 2e^{\frac{x}{2}} + e^{\frac{3x}{2}} + 1) dx$$

$$= \left[\frac{-x^2}{4} - 4e^{\frac{x}{2}} + \frac{2}{3}e^{\frac{3x}{2}} + x \right]_{-2}^{0}$$ **(1 mark)**

$$= \left\{ (0 - 4e^0 + \frac{2}{3}e^0 + 0) - (\frac{-4}{4} - 4e^{-1} + \frac{2}{3}e^{-3} - 2) \right\}$$

$$= -4 + \frac{2}{3} + 1 + 4e^{-1} - \frac{2}{3}e^{-3} + 2$$

$$= \frac{-1}{3} + \frac{4}{e} - \frac{2}{3e^3}$$ square units **(1 marks)**

Total 15 marks