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MATHEMATICAL METHODS TRIAL EXAM 1 2000 SOLUTIONS

Part I - Multiple choice answers

1.	A	8.	E	15.	C	22.	E
2.	C	9.	C	16.	B	23.	D
3.	C	10.	A	17.	C	24.	D
4.	D	11.	C	18.	B	25.	B
5.	D	12.	D	19.	A	26.	D
6.	B	13	D	20.	B	27.	D
7.	D	14.	E	21.	C	28.	B

Part I - Multiple choice solutions

Question 1

Mean is given by $\mu = \Sigma x\, p(x)$

$$= 0 \times 0.4 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.1$$
$$= 1$$

Variance is given by $\sigma^2 = \Sigma x^2\, p(x) - \mu^2$

$$= 0 \times 0.4 + 1 \times 0.3 + 4 \times 0.2 + 9 \times 0.1 - 1^2$$
$$= 1$$

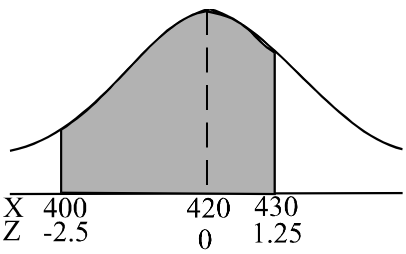
The answer is A.

Question 2

In general, discrete variables can be counted, for example, the number of goals, the scores on a test, the number of people comprising a population, the number of cents making up a price, and so on. Things which are measured like height and weight are not discrete; rather they are continuous. The answer is C.

Question 3

We are seeking to find the shaded area.



Now, $z = \dfrac{X - \mu}{\sigma}$

So, $z = \dfrac{400 - 420}{8}$

$= \dfrac{-20}{8}$

$= -2.5$

Also $z = \dfrac{430 - 420}{8}$

$= 1.25$

We need to find $\Pr(-2.5 < Z < 1.25) = \Pr(Z < 1.25) - \Pr(Z < -2.5)$

$$= \Pr(Z < 1.25) - (1 - \Pr(Z < 2.5))$$
$$= 0.8944 - (1 - 0.9938)$$
$$= 0.8882$$

The proportion of buns weighing between 400 and 430 grams is closest to 89%.
The answer is C.

Question 4

The mean of the distribution in option A is not zero. It appears that over 90% of the distribution in option B lies 1 standard deviation either side of the mean. For Z, there should be about 68% of the distribution between 1 standard deviation either side of the mean. Option C shows a curve which is not bell-shaped. Option D is a feasible option. Option E has about 90% or more of the distribution between values of -20 and 20. This is not feasible for the standard normal distribution. The answer is D.

Question 5

If X is the number of red lollies in the bag, then X has a hypergeometric distribution since the lollies are taken out and not returned, that is, there is no replacement.

So, $\Pr(X = x) = \dfrac{{}^{D}C_{x}\,{}^{N-D}C_{n-x}}{{}^{N}C_{n}}$ where $x = 6,\ D = 80,\ N = 200,\ n = 10$

$\Pr(X = 6) = \dfrac{{}^{80}C_{6}\,{}^{120}C_{4}}{{}^{200}C_{10}}$ The answer is D.

Question 6

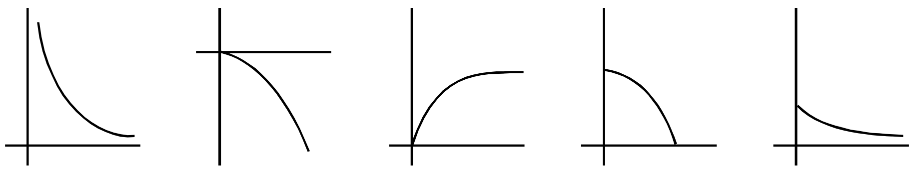
Let X equal the number of times Year 12s are present at the next 3 detention classes. Now, X follows a binomial distribution with $n = 3,\ p = 0.15$

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X = 0)\\ &= 1 - {}^{3}C_{0}(0.15)^{0}(0.85)^{3}\end{aligned}$$

The answer is B.

Question 7

The graphs of the options are as follows.



$y = \dfrac{a}{x^2}$ $y = -ax^2$ $y = ax^{\frac{1}{2}}$ $y = \cos(ax)$ $y = e^{-ax}$

The only possibility is option D. The answer is D.

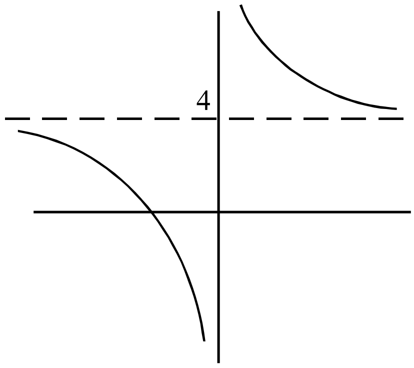
Question 8

The equation of the asymptote for the original function is y = -2 The equation of the asymptote for the inverse function is therefore *x* = -2 This eliminates A, B and C. The original function passes through the point (0, -1). The inverse function will therefore have to pass through the point (-1, 0). The graph of the inverse function is a reflection of the original function in the line y = *x* The answer is E.

Question 9

For the graph with equation $y = \dfrac{A}{x+b} + B$, the equation of the vertical asymptote is given by *x* = -*b* and the equation of the horizontal asymptote is given by *y* = *B*. The answer is C.

Question 10



Sketch the graph on your graphics calculator.
We see that the domain is $(-\infty, 0) \cup (0, \infty)$ and the range is $(-\infty, 4) \cup (4, \infty)$.
Only option A offers this combination.
The answer is A.

Question 11

For $x \in (-5, -1]$ the gradient function of g; that is, g' is constant and is given by $\frac{\text{rise}}{\text{run}} = \frac{4 - -4}{-1 - -5}$

$$= \frac{8}{4} = 2$$

For $x \in (-1, 1]$, the gradient of g is zero, that is, $g' = 0$
For $x \in (1, 6)$, the gradient is negative but becoming less steep and as x approaches 6, the gradient approaches zero. Only options C and E show all of this. Since g is not continuous at x =-1, the derivative does not exist at x = -1. So option E is incorrect. The answer is C.

Question 12

$$f(x) = x^2 \tan(4x)$$

So, $f'(x) = 2x\tan(4x) + x^2 4\sec^2(4x)$

$$= 2x\tan(4x) + 4x^2 \sec^2(4x)$$

The answer is D.

Question 13

Now, $y = \frac{e^{\sin(3x)}}{2}$ Let $u = \sin(3x)$ and so $\frac{du}{dx} = 3\cos(3x)$

So, $y = \frac{e^u}{2}$

and $\frac{dy}{du} = \frac{e^u}{2}$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (chain rule)

$$= \frac{e^u}{2} 3\cos(3x)$$

$$= \frac{e^{\sin(3x)}}{2} 3\cos(3x)$$

$$= \frac{3}{2}\cos(3x)e^{\sin(3x)}$$

The answer is D.

Question 14

$$\frac{d\left(\frac{\log_e(x^2-4x)}{x}\right)}{dx} = \frac{x.\frac{2x-4}{x^2-4x} - \log_e(x^2-4x)\times 1}{x^2}$$

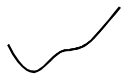
$$= \frac{\frac{x(2x-4)}{x(x-4)} - \log_e(x^2-4x)}{x^2}$$

$$= \frac{2x-4}{x^2(x-4)} - \frac{1}{x^2}\log_e(x^2-4x)$$

The answer is E.

Question 15

We can find the stationary points by solving $\frac{dy}{dx} = 0$, that is, $4x^3 + 24x^2 + 10x + 1 = 0$

It is faster, however, to use a graphics calculator and look at the graph of $y = x^4 + 8x^3 + 5x^2 + x$ to see the points where the gradient is zero. We know that the general shape of a "positive quartic" is  If the graph has a point of inflection it could look like this 

Adjust the WINDOW on your graphics calculator so that all turning points are included. This means looking at some very low values of *y*. Find the minimum turning points of the function using your graphics calculator. They occur at (0, 0) and approximately (-5.558, -270.378).
The answer is C.

Question 16

The graph of *g* should be the derivative graph of the A - E graphs. By taking each option in turn, we can eliminate those which are incorrect.
The gradient of graph A for $x \in (-2, 0)$ and $x \in (0, 3)$ is zero. This eliminates option A.
For option B, for $x \in (-2, 0)$ the gradient is -1 and for $x \in (0, 3)$, the gradient is positive but decreasing. This is reflected in the graph of *g*. None of the options C, D or E have a derivative function described by function *g*. The answer is B.

Question 17

Now, $\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + c_1 \qquad n \neq -1 \qquad c_1$ is a constant

So, $\int (2x+1)^{-2} = \frac{(2x+1)^{-1}}{2\times -1} + c_1$

$$= \frac{1}{-2(2x+1)} + c_1$$

Also, $\int \sin(ax)dx = -\frac{1}{a}\cos ax + c_2 \qquad c_2$ is a constant

So, $\int \sin\frac{x}{2}dx = \frac{-1}{\frac{1}{2}}\cos\frac{x}{2} + c_2$

$$= -2\cos\frac{x}{2} + c_2$$

So, $\int(\sin\frac{x}{2} + \frac{1}{(2x+1)^2})dx = -2\cos\frac{x}{2} + \frac{1}{-2(2x+1)} + c$ where $c = c_1 + c_2$

$$= -2\cos\frac{x}{2} - \frac{1}{2(2x+1)} + c$$

The answer is C.

Question 18

The area of the "left rectangles" is given by "width $\times$ height"+"width $\times$ height"

$$= 1\times f(1) + 1\times f(2)$$
$$= 1\times(-(1)^2 + 10) + 1\times(-(2)^2 + 10)$$
$$= 9 + 6 = 15$$

The area of the "right rectangles" is given by "width $\times$ height"+"width $\times$ height"

$$= 1\times f(2) + 1\times f(3)$$
$$= 1\times(-(2)^2 + 10) + 1\times(-(3)^2 + 10)$$
$$= 6 + 1 = 7$$

The average of these two values is $\frac{15+7}{2} = 11$. The approximate value is 11. The answer is B.

Question 19

For $x \in [-3, 0]$, $f(x) > g(x)$. Therefore the shaded area shown in this domain is given by

$$\int_{-3}^{0}(f(x) - g(x))dx$$

For $x \in [0, 2]$, the shaded area under the curve is given by $-\int_{0}^{2} f(x)dx$.

Note that $\int_{0}^{2} f(x)dx$ equals a negative number and since we are seeking an actual area, we must multiply the integral by -1.

The total area is therefore given by $\int_{-3}^{0}(f(x) - g(x))dx - \int_{0}^{2} f(x)dx$

The answer is A.

Question 20

For f the amplitude is 3 and the period is given by $\frac{2\pi}{n}$.

Now, $f(x) = 3\sin 4(x - \frac{\pi}{2}) + 1$ So, $n = 4$ So the period is $\frac{2\pi}{4} = \frac{\pi}{2}$. The graph of $y = \sin x$ has been translated $\frac{\pi}{2}$ units to the right. The answer is B.

Question 21

From the graph we see that the amplitude is 2. This eliminates options D and E. The period is $\frac{\pi}{2}$ and period = $\frac{2\pi}{n}$. So, $\frac{2\pi}{n} = \frac{\pi}{2}$ and so $n = 4$. This eliminates options A and B.

The graph is a sin graph translated $\frac{\pi}{4}$ units left. Only option C reflects all this.

The answer is C.

Question 22

Now, $10\cos(3x) = 5$

$$\cos(3x) = \frac{5}{10}$$

$$\cos(3x) = \frac{1}{2}$$

So, $$3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$$

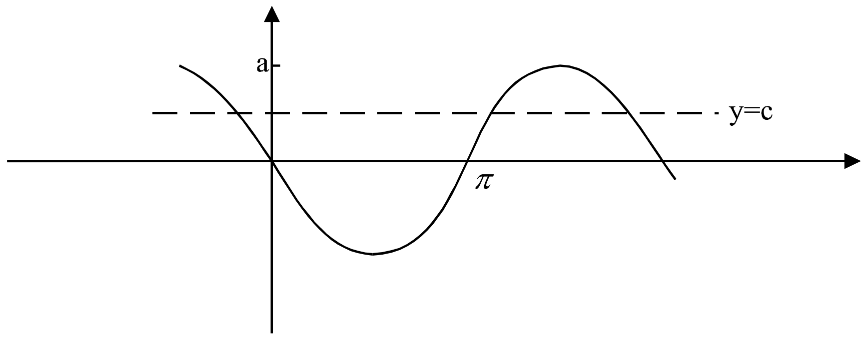
$$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \ldots$$

Over the domain $[0, \pi]$, $x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

The sum of these solutions is $\frac{13\pi}{9}$. The answer is E.

Question 23

$f(x) = c$ occurs when the graph of *f(x)* intersects with the line with equation $y = c$. Since $0 < c < a$, this line is a horizontal line which lies above the *x* axis and below the horizontal line with equation $y = a$.
In the interval $[0, \pi]$, there will be no intersection and therefore $f(x) = c$ will have no solution if the graph of *f(x)* appears on or below the *x* axis.
This will only happen if *b* equals π.



The answer is D.

Question 24

The graph shown is that of a polynomial of degree 4 not 3. This eliminates options A and C. The graph touches the *x* axis at *x* = 0 rather than cutting it as it does at *x* = a and *x* = b. Therefore there is a repeated root of *x* = 0. The other two roots are *x* = *a* and *x* = *b*. The corresponding factors are *x, x, x - a* and *x - b*. The graph is inverted, that is, the general shape of a positive quartic (which has a $+x^4$ term) 

is inverted to become 

which is the shape of a negative quartic (which has a $-x^4$ term)
Therefore the x^4 term of our polynomial will have a negative coefficient. The required equation is $y = -x^2(x-a)(x-b)$. Choose values of *a* and *b* and check the shape of the graph on your graphic calculator. The answer is D.

Question 25

Expand $(3x-a)^6$ using Pascals triangle to obtain the coefficients of the terms.

We have $(3x-a)^6 = 1\times(3x)^6\times(-a)^0 + 6\times(3x)^5\times(-a)^1 + 15\times(3x)^4\times(-a)^2 + ...$

The x^4 term is given by $15\times(3x)^4\times(-a)^2 = 4860x^4$

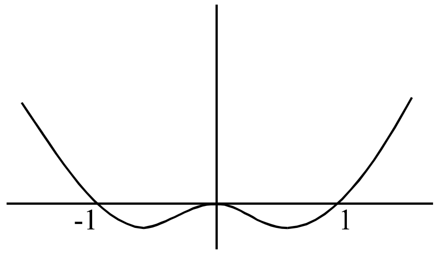
So, $$15\times 3^4\times a^2 = 4860$$
$$a = \pm 2$$

Reread the question. We note that $a > 0$. So *a* = 2.
The answer is B.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Question 26

Sketch the graph using a graphic calculator
or by using the fact that $f(x) = x^4 - x^2$
$= x^2(x^2 - 1)$
$= x^2(x-1)(x+1)$



For an inverse to exist, we require that the function *f* over the domain $[a, b]$ is one : one.
That is, a horizontal line can be drawn at any point on the function over that domain and will cut the function at one point only.
Note that $f'(x) = 4x^3 - 2x = 0$ gives $2x(2x^2 - 1) = 0$ and so the turning points occur at $x = 0$ and when $2x^2 - 1 = 0$, that is, $x^2 = \frac{1}{2}$ and so $x = \pm\frac{1}{\sqrt{2}}$.
For $x \in [-1, 0]$, the function is not 1 : 1
For $x \in [-1, 1]$, the function is not 1 : 1
For $x \in [0, 1]$, the function is not 1 : 1
For $x \in [-\frac{1}{\sqrt{2}}, 0]$, the function <u>is</u> 1 : 1
For $x \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, the function is not 1 : 1
The answer is D.

Question 27

Now, $d_g = [1, \infty)$. By graphing *g(x)*, we see that $r_g = [3, \infty)$ and so $d_{g^{-1}} = [3, \infty)$

Consider $g(x) = 2(x-1)^2 + 3$

Let $y = 2(x-1)^2 + 3$

Swap *x* and *y*. $x = 2(y-1)^2 + 3$

Rearranging: $\frac{x-3}{2} = (y-1)^2$

$y - 1 = \pm\sqrt{\frac{x-3}{2}}$

So $y = \sqrt{\frac{x-3}{2}} + 1$ Choose the upper branch. The graph of $y = -\sqrt{\frac{x-3}{2}} + 1$ would be obtained if we were finding the inverse of the function *f* say, where $f : (-\infty, 1] \rightarrow R$ where $f(x) = 2(x-1)^2 + 3$, that is, the other branch

The answer is D.

Question 28

$$\frac{1}{2}\log_2 2 - \log_2 \sqrt{8} - \log_2 x = 1$$

So, $$\frac{1}{2}\times 1 - \log_2 \sqrt{8} - \log_2 x = 1$$

$$-\log_2 \sqrt{8} - \log_2 x = \frac{1}{2}$$

$$\log_2 \frac{1}{\sqrt{8}\times x} = \frac{1}{2}$$

$$\log_2 \frac{1}{2\sqrt{2}x} = \frac{1}{2}$$

So, $$2^{\frac{1}{2}} = \frac{1}{2\sqrt{2}x}$$

$$\sqrt{2} = \frac{1}{2\sqrt{2}x}$$

So, $$4x = 1$$

$$x = \frac{1}{4}$$

The answer is B.

Alternative Method

$$\frac{1}{2}\times 1 - \log_2 2^{\frac{3}{2}} - \log_2 x = 1$$

$$\frac{1}{2} - \frac{3}{2} - \log_2 x = 1$$

$$-1 - \log_2 x = 1$$

$$-\log_2 x = 2$$

$$\log_2 x = -2$$

$$2^{-2} = x$$

$$x = \frac{1}{4}$$

PART II - Short answer solutions

Question 1

$$x^4 - 6x^3 - x^2 + 6x = x(x^3 - 6x^2 - x + 6)$$
$$= x(x-1)(x+1)(x-6)$$

Let $p(x) = x^3 - 6x^2 - x + 6$

So, $p(1) = 1 - 6 - 1 + 6 = 0$

So, $(x-1)$ is a factor

So, $p(x) = (x-1)(x^2 - 5x - 6)$
$= (x-1)(x-6)(x+1)$

(1mark) for first 2 factors and **(1 mark)** for second 2 factors

Question 2

a. Since we have a discrete probability distribution we know that

$$\frac{2}{q} + \frac{7}{2q} + 0.2 + \frac{3}{2q} + \frac{1}{q} = 1$$

So, $$\frac{10}{2q} + \frac{3}{q} = 0.8$$

$$\frac{5}{q} + \frac{3}{q} = 0.8$$

$$\frac{8}{q} = 0.8$$

$$q = \frac{8}{0.8}$$

$$= 10$$ **(1 mark)**

b. $E(x) = \Sigma xp(x)$

$$= 0\times\frac{2}{10} + 1\times\frac{7}{20} + 2\times 0.2 + 3\times\frac{3}{20} + 4\times\frac{1}{10}$$

$$= 1.6$$ **(1 mark)**

c. $\Pr(X \ge 1) = \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4)$

$= 1 - \Pr(X = 0)$

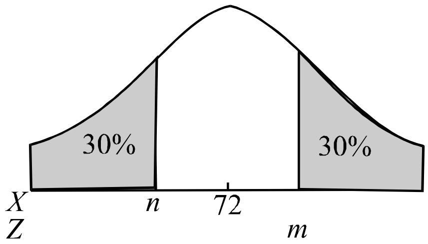
$= 1 - 0.2$

$= 0.8$ **(1 mark)** The answer $\Pr(X \ge 1) = 1 - \frac{2}{q}$ is acceptable if q was not found in part **a.**

Question 3

Draw a diagram.

The shaded area to the left of n represents 30% of the area under the bell shaped curve, that is, $\Pr(X < n) = 0.30$.



We need to find the corresponding point to n above the mean. Let us call this point m.

Using the standard normal function, we have

$\Pr(Z < m) = 70\%$

From the tables we have $m = 0.524$ **(1 mark)**

Now, $m = \dfrac{X - \mu}{\sigma}$

So, $0.524 = \dfrac{X - 72}{5}$

$X = 74.62$

Now, $74.62 - 72 = 2.62$

So, $n = 72 - 2.62$ (because of the symmetry of the bell shaped curve)

$= 69.38$ minutes So, in 30% of races she has been in, Kerry has taken less than 69.38 minutes. The value of n is 69.38 **(1 mark)**

Question 4

a. The value of the gradient at $x = 1.2$ can be found by numerical differentiation on a graphics calculator. The value is 4.583 to 3 decimal places. **(1 mark)**

b. Now, $10^{0.65} = 10^{\frac{1.3}{2}} = 10^{\frac{1.2+0.1}{2}}$ and $f(x+h) \approx f(x) + hf'(x)$ **(1 mark)**

From part **i.** $f'(x) = f'(1.2) = 4.583$

Also, $h = 0.1$ and $f(1.2) = 10^{\frac{1.2}{2}} = 3.981$ (to 3 decimal places)

So, $f(x + 0.1) \approx 3.981 + 0.1 \times 4.583$

$= 4.44$ correct to 2 decimal places **(1 mark)**

Question 5

a. Average rate of change between $t = 0$ and $t = 3$ is given by $= \dfrac{v(3) - v(0)}{3}$

$= \dfrac{3.15129 - 2}{3}$

$= 0.4 \text{ m/s}^2$ to 1 decimal place

(1 mark)

b. To find the instantaneous rate of change at time $t = 3$ secs, we need to first find $\dfrac{dv}{dt}$

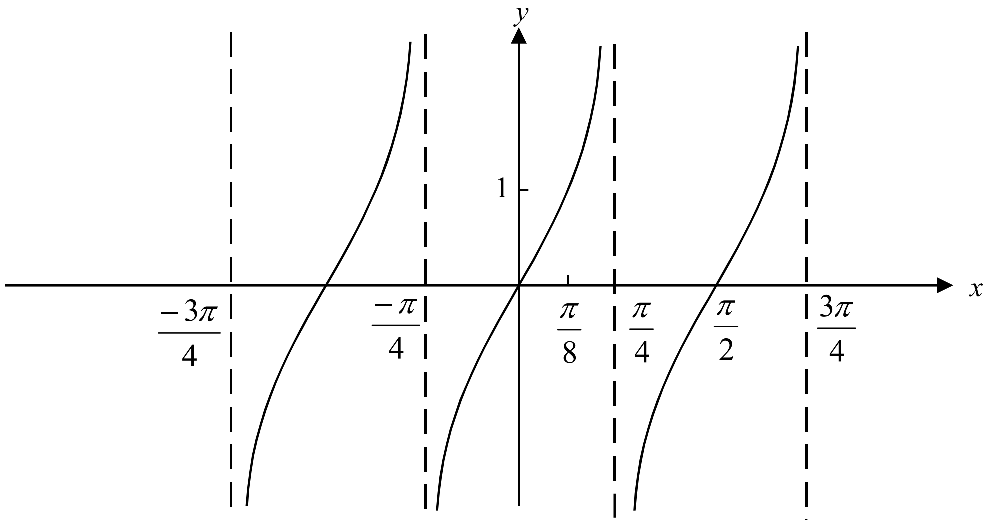
Now, $v = 0.5\log_e(t^2 + 1) + 2$

So, $\dfrac{dv}{dt} = 0.5 \times \dfrac{2t}{t^2 + 1}$

$= \dfrac{t}{t^2 + 1}$ **(1 mark)** At $t = 3$, $\dfrac{dv}{dt} = 0.3 \text{ m/s}^2$ **(1 mark)**

Question 6

a. The period of this graph is given by $\frac{\pi}{n} = \frac{\pi}{2}$. So the asymptotes closest to the *y* axis will occur at $x = \frac{\pi}{4}$ and $x = -\frac{\pi}{4}$ The next will occur at $\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ and $\frac{-\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}$
Draw the asymptotes and then draw the tan graph.



(2 marks)

b. Now, $\sqrt{3}\sin 2x = \frac{3}{\sqrt{3}}\cos 2x$

$$\sin 2x = \frac{3}{3}\cos 2x$$

$$\frac{\sin 2x}{\cos 2x} = 1$$

$$\tan 2x = 1$$

$$2x = \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = \frac{-3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$$

Check using the graph. The graph of $y = \tan 2x$ intersects the line with equation $y = 1$ at 3 points over the domain $(\frac{-3\pi}{4}, \frac{3\pi}{4})$. Those points occur when $x = \frac{-3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$ **(1 mark)**

Question 7

$$y = 2x^2 + 1$$

$$\frac{dy}{dx} = 4x$$

At $x = 2$, gradient of tangent is 8.
At $x = 2, y = 9$
Equation of tangent is $y - 9 = 8(x - 2)$

$$y = 8x - 7$$ **(1 mark)**

Now, $y = \sqrt{x-2}$

$= (x-2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x-2)^{\frac{-1}{2}} \times 1$

$= \frac{1}{2\sqrt{x-2}}$

At $x = 3$, gradient of tangent is $\frac{1}{2}$. So at $x = 3$, gradient of normal is -2 At $x = 3$, $y = 1$

Equation of normal is $y - 1 = -2(x-3)$

$y = -2x + 7$ **(1 mark)**

Now, $y = 8x - 7$

and $y = -2x + 7$

So, $8x - 7 = -2x + 7$

$10x = 14$

$x = 1.4$

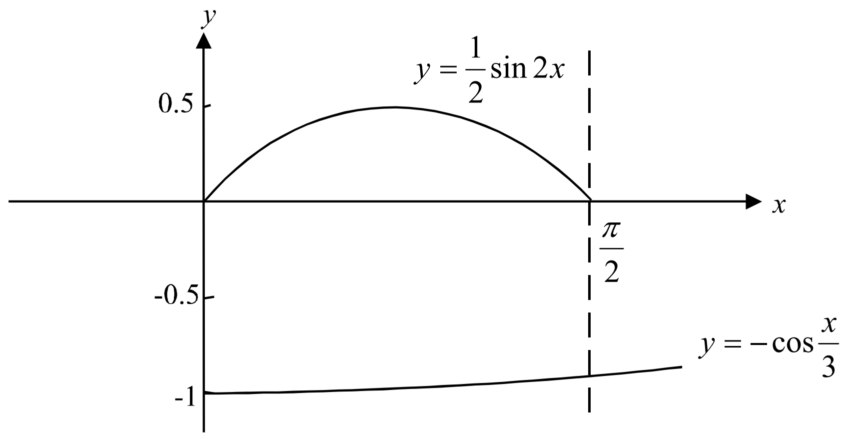
Substitute into $y = 8x - 7$

So, $y = 4.2$

Point of intersection of tangent and normal is (1.4, 4.2). **(1 mark)**

Question 8

Do a quick sketch.



Area required $= \int_0^{\frac{\pi}{2}} (\frac{1}{2}\sin(2x) + \cos\frac{x}{3})dx$ **(1 mark)**

$= \left[\frac{-1}{4}\cos(2x) + 3\sin\frac{x}{3}\right]_0^{\frac{\pi}{2}}$ **(1 mark)**

$= \left\{(\frac{-1}{4} \times \cos\pi + 3\sin\frac{\pi}{6}) - (\frac{-1}{4}\cos 0 + 3\sin 0)\right\}$

$= \left\{(\frac{1}{4} + \frac{3}{2}) - (\frac{-1}{4} + 0)\right\}$

$= \frac{7}{4} + \frac{1}{4}$

$= 2$ **(1 mark)**

Total 22 marks