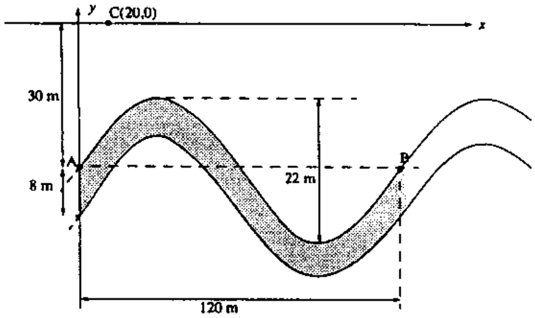
Question 1

All answers in this question should be given to 2 decimal places.

A layer of ore beneath the ground in outback Australia has surfaces that are sinusoidal in cross section. A miner has drawn up a rough sketch on a set of axes has been set up so that the x-axis represents the surface:



The miner feels that an appropriate equation to represent the top level of the sinusoidal curve, relative to the set of axes that she has drawn is

$$y = a \times \sin(kx) + d,\ 0 \le x \le 120.$$

a. Show that $a = 11$, $d = -30$ and $k = \frac{\pi}{60}$

Amplitude $= \frac{1}{2} \times 22 = 11 \quad \therefore a = 11$ M1

From $A(0,-30)$: $-30 = a\sin(0) + d \Rightarrow d = -30$ M1

Period $= 120 \quad \therefore \frac{2\pi}{k} = 120 \Rightarrow k = \frac{\pi}{60}$ M1

[3 marks]

b. What is the minimum distance that miners will need to drill to reach the ore?

From 'graph': min distance $= 30 - 11$

$= 19$ m A1

[1 mark]

c. How deep must miners drill when they are at point C, if they must drill through the layer?

$x = 20,\ y = 11\sin\left(20 \times \frac{\pi}{60}\right) - 30$

$= 11 \times \frac{\sqrt{3}}{2} - 30$ M1

$\therefore$ needs to drill $\left|\left(11 \times \frac{\sqrt{3}}{2} - 30\right) - 8\right| \simeq 28.47$ m A1

[2 marks]

Once the miners have drilled at point C and have reached the lower layer, a second pipe is to run horizontally until it hits the layer of ore once again.

d. What is the minimum possible length for this horizontal pipe?

$11 \times \frac{\sqrt{3}}{2} - 30 - 8 = 11\sin\left(\frac{\pi}{60}x\right) - 30 - 8$ M1

$\sin\left(\frac{\pi}{60}x\right) = \frac{\sqrt{3}}{2}$

$\frac{\pi}{60}x = \frac{\pi}{3}, \frac{2\pi}{3}$ M1

$\therefore x = 20, 40$ A1

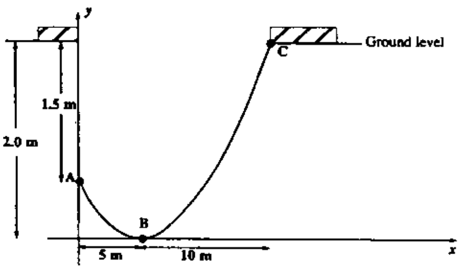
$\therefore$ minimum length is 20 m. A1

[4 marks]

Total 10 marks

Question 2

The cross sectional area of a swimming pool (shown below) has been set up on a set of axes and is modelled by the equation $y = a(x-b)^2 + c,\ 0 \le x \le 15$.



a. i. Write down the coordinates of the A, B and C.

$A(0, \frac{1}{2})$, $B(5,0)$ & $C(15,2)$ A1

ii. Hence, show that $a = \frac{1}{50}$, $b = 5$ and $c = 0$.

Turning pt at $(5,0)$ $\therefore b = 5$ & $c = 0$ M1, M1

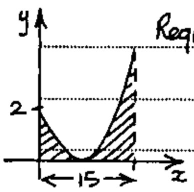
$\therefore y = a(x-5)^2$

When $x = 0$, $y = \frac{1}{2}$ $\therefore \frac{1}{2} = a(-5)^2$ M1

$\Leftrightarrow a = \frac{1}{50}$

[4 marks]

b. i. Find the cross sectional area of the water in the pool when the pool is full.



Required area $= 2 \times 15 - \int_0^{15} \frac{1}{50}(x-5)^2\,dx$ M1, M1

$= 30 - \frac{1}{150}\left[(x-5)^3\right]_0^{15}$ A1

$= 30 - \frac{1125}{150}$

$= 22.5\ m^2$ A1

ii. Given that the swimming pool has a constant length of 10 m for all depths of water, find the volume of water when the swimming pool is full.

Volume $= 10 \times 22.5$

$= 225\ m^3$ A1

[5 marks]

c. i. Find the value of x when the depth of water in the pool is 1.5 metres.

$\frac{1}{50}(x-5)^2 = 1.5$ M1

$\Leftrightarrow (x-5)^2 = 75$

$\Leftrightarrow x - 5 = \pm 8.66$

But $0 \le x \le 15$, $\therefore x = 13.66$ A1

ii. Show that the value of x when the depth of water is h metres, where $h > 0.5$, is given by $5(1+\sqrt{2h})$.

$\frac{1}{50}(x-5)^2 = h$

$\Leftrightarrow \quad (x-5)^2 = 50h$

$\Leftrightarrow \quad x-5 = \pm 5\sqrt{2h}$ M1

As $x \in [0,15]$, $x = 5 + 5\sqrt{2h}$

$= 5(1+\sqrt{2h})$

[3 marks]

d. Show that the cross sectional area of water in the swimming pool, $A\ m^2$, when the water surface reaches a height of h metres, where $h > 0.5$, is given by

$$A = \left(5h + \frac{10}{3}h\sqrt{2h} - \frac{5}{6}\right)$$

$A = h(5+5\sqrt{2h}) - \int_0^{5+5\sqrt{2h}} \frac{1}{50}(x-5)^2\,dx$

$= 5h + 5h\sqrt{2h} - \frac{1}{150}\left[(x-5)^3\right]_0^{5+5\sqrt{2h}}$

$= 5h + 5h\sqrt{2h} - \frac{1}{150}\left(250h\sqrt{2h} + 125\right)$

$= 5h + 5h\sqrt{2h} - \frac{5}{3}h\sqrt{2h} - \frac{5}{6}$

$= 5h + \frac{10}{3}h\sqrt{2h} - \frac{5}{6}$

[4 marks]

e. Using the fact that the length of the pool is 10 m and the result from part d., find the rate of change of volume with respect to height, when the depth of water is 2 metres.

$V = 10\left(5h + \frac{10}{3}h\sqrt{2h} - \frac{5}{6}\right)$

$= 50h + \frac{100}{3}\sqrt{2}\,h^{3/2} - \frac{25}{3}$

$\frac{dV}{dh} = 50 + 50\sqrt{2}\,h^{1/2}$ M1

$= 50 + 50\sqrt{2h}$

$h=2$, $\frac{dV}{dh} = 50 + 50 \times 2$

$= 150$ A1

$\therefore$ Volume is increasing at $150\ m^3/m$.

[2 marks]

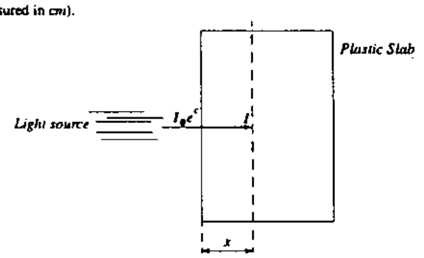
Total 18 marks

Question 3

As light passes through a slab of plastic, its intensity, I units, decreases according to the model

$$I = I_0 \times e^{-kx+c}, x \geq 0$$

where I_0e^c is the intensity of light at the surface of the slab (at $x = 0$) and x is the depth of penetration of the light (measured in cm).



Different plastics are being tested in the hope that such slabs will prove to be cheaper to produce than glass when used in the constructing fish tanks. The set of results shown in Table 1, was recorded for a particular slab labelled 'Slab A'.

Table 1:

Depth of penetration x cm	Intensity I units
0.2	$0.9I_0$
1.2	$0.8I_0$

a. Using the results of Table 1 and the model $I = I_0 \times e^{-kx+c}$, $x \geq 0$, show that

i. $c - 0.2k = \log_e(0.9)$

$x = 0.2$, $I = 0.9I_0 \quad \therefore\ 0.9 = e^{-0.2k+c}$ M1

$\Leftrightarrow -0.2k + c = \log_e 0.9$ M1

— ①

ii. $c - 1.2k = \log_e(0.8)$

$x = 1.2$, $I = 0.8I_0 \quad \therefore\ 0.8 = e^{-1.2k+c}$ M1

$\Leftrightarrow -1.2k + c = \log_e 0.8$ M1

— ②

[4 marks]

b. Hence show that $k = \log_e\frac{9}{8}$ and $c = \log_e 0.9 + 0.2\log_e\frac{9}{8}$.

① − ② : $1.0k = \log_e(0.9) - \log_e(0.8) = \log_e\left(\frac{9}{8}\right)$ M1

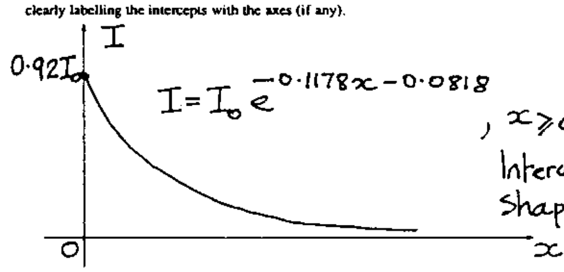
sub into ① : $-0.2\log_e\left(\frac{9}{8}\right) + c = \log_e(0.9)$

$\therefore\ c = \log_e(0.9) + 0.2\log_e\left(\frac{9}{8}\right)$ M1

[2 marks]

From part b., we have that $k = 0.1178$ and $c = -0.0818$. Use these results for parts c., to e.

c. On the set of axes provided, sketch the graph of the intensity I versus the depth of penetration x clearly labelling the intercepts with the axes (if any).



Intercept A1
Shape A1

[2 marks]

d. If this slab of plastic has a total width of 2.1 cm, what will the intensity of the light be once it reaches the water inside the tank. Give your answer as a percentage of I_0, to the nearest percent.

$x = 2.1$, $I = I_0 e^{-0.1178 \times 2.1 - 0.0818}$ M1

$= 0.7195 I_0$

i.e., approximately 72% of I_0 A1

[2 marks]

It has been decided that 85% of the intensity of light should be sufficient to provide a bright enough environment for the fish in a tank that has been constructed using Slab A.

e. How thick should a piece of Slab A type material be in order to meet the new '85% criteria'? Give your answer to the nearest *mm*.

$0.85 I_0 = I_0 e^{-0.1178x - 0.0818}$ M1

$\log_e(0.85) = -0.1178x - 0.0818$ M1

$\Leftrightarrow \quad x = 0.6852$

i.e., 7mm A1

[3 marks]

A 'Super resin' is added during the production of slab A to 'strengthen' the plastic. Its overall effect on the light intensity is to reduce the intensity at a faster rate as it passes through the plastic. This new material is labelled 'Slab B'.

Unfortunately, when slabs of type B go beyond a certain width the 'Super resin' can no longer reduce the light intensity. In fact it is found that after light penetrates a certain distance, the intensity starts to increase. It is thought that an appropriate model to represent this situation is given by

$$I = I_0 - xe^{-kx+c}, x \geq 0,$$

where I_0 is the intensity of light at the surface of the slab (at $x = 0$) and x is the depth of penetration of the light inside the slab of type B plastic.

f. Show that the minimum intensity reached by light passing within slabs of type B occur at $x = \frac{1}{k}$ and hence find the minimum intensity found within the slab.

$I'(x) = -\left(e^{-kx+c} - kxe^{-kx+c}\right)$

$= -(1-kx)e^{-kx+c}$ A1

$I'(x) = 0 \Leftrightarrow -(1-kx)e^{-kx+c} = 0$ M1

$\Leftrightarrow \quad x = 1/k$ A1

At $x = \frac{1}{k}$, $I = I_0 - \frac{1}{k}e^{-k \times \frac{1}{k} + c}$

$= I_0 - \frac{1}{k}e^{-1+c}$

$= I_0 - \frac{1}{k}e^{c-1}$ A1

Use of sign test (of first derivative) M1

[5 marks]

Total 18 marks

Question 4

All answers in this question should be given to 4 decimal places.

Don, the school photographer has found that the time, T seconds, taken to develop prints may be described as a random variable that is normally distributed with a mean of 16.5 seconds and a variance of 0.50 seconds.

a. Find the probability that the time taken to develop the next print is

i. at most 16 seconds,

$Pr(T \leq 16) = Pr(Z \leq -0.7071)$

$= 1 - Pr(Z \leq 0.7071)$

$= 0.2399$ A1

ii. at least 16.7 seconds,

$Pr(T \geq 16.7) = Pr(Z \geq 0.2828)$

$= 1 - Pr(Z \leq 0.2828)$

$= 0.3885$ A1

iii. between 16 and 16.7 seconds,

$Pr(16 \leq T \leq 16.7) = 1 - (0.2399 + 0.3885)$

$= 0.3716$ A1

iv. at least 16.5 seconds given that it took between 16 and 16.7 seconds.

$Pr(T \geq 16.5 \mid 16 \leq T \leq 16.7) = \dfrac{Pr(16.5 \leq T \leq 16.7)}{Pr(16 \leq T \leq 16.7)}$ M1

$= \dfrac{0.6115 - 0.5}{0.3716}$

$= 0.3001$ A1

[5 marks]

Don needs to develop six such prints.

b. i. Find the probability that the first two prints take no more than 16 seconds and that the next four take at least 16.7 seconds.

$(0.2399)^2 \times (0.3885)^4 = 0.0013$ M1 A1

ii. Find the probability that only two of the six prints will take no more than 16 seconds to print.

${}^6C_2\,(0.2399)^2 \times (0.7601)^4 = 0.2882$ M1 A1

iii. Find the probability that at least two of the prints will take between 16 seconds and 16.7 seconds.

Let N denote the no. of prints $\therefore N \stackrel{d}{=} Bi(6, 0.3716)$ M1

$Pr(N \geq 2) = 1 - \left[0.6284^6 + {}^6C_1(0.6284)^5(0.3716)\right]$ M1

$= 0.7199$ A1

[7 marks]

c. i. How long could Don expect to take to develop 200 prints?

$200 \times 16.5 = 3300$ sec (55 mins) A1

ii. Give an approximate 95% confidence interval for the time taken to develop a single print.

$16.5 \pm 2 \times 0.7071$ M1

$15.1 \leq T \leq 17.9$ A1

[3 marks]

Don buys a new supply of chemicals that are guaranteed to decrease the time spent in developing prints. In fact, the new time, X seconds, it given by

$$X = aT \text{ where } 0 < a < 1.$$

d. i. Show that $E(X) = 16.5a$ and $Var(X) = 0.5a^2$

$E(aT) = aE(T) = 16.5a$

$Var(aT) = a^2 Var(T) = 0.5a^2$ M1

ii. Given that the probability of taking more than 16.5 seconds is reduced by 20%, find the value of a.

$Pr(X \geq 16) = 0.8 Pr(T \geq 16) = 0.6081$ M1

$\therefore Pr(X \leq 16) = Pr\left(Z \leq \frac{16 - 16.5a}{a\sqrt{0.5}}\right) = 0.3919$ M1

$\frac{16 - 16.5a}{a\sqrt{0.5}} = -0.275$ A1

$\therefore a = 0.9813$ A1

[5 marks]

Total 20 marks