1995
VCE
MATHEMATICAL METHODS
CAT 3

DETAILED SUGGESTED SOLUTIONS

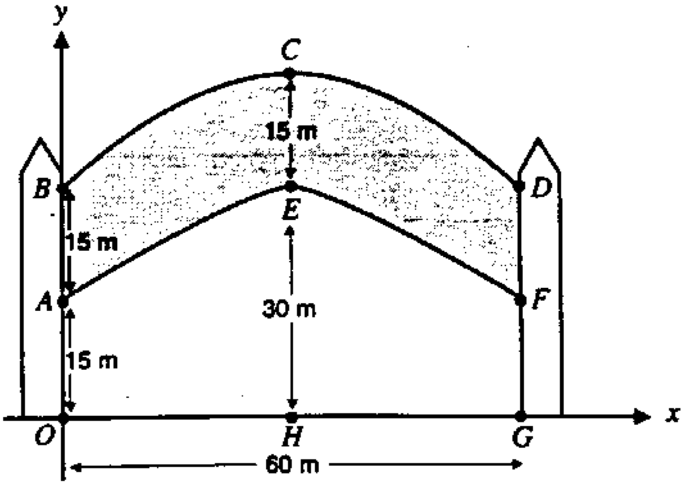
CHEMISTRY ASSOCIATES
P.O. BOX 2227
KEW, VIC., 3101
AUSTRALIA
TEL:(03) 9817 5374 FAX: (03) 9817 4334

email: chemas@vicnet.net.au

Internet: http://www.vicnet.net.au/~chemas/education.htm

Suggested Solutions: 1995 VCE Mathematical Methods CAT 3

Question 1



A newly designed bridge is to be built across the 60 metre Yarra River. The architect draws on a set of axes a cross-section of the bridge, which is symmetrical about the line CH, as shown in the diagram above. The water surface is along OG. The architect introduces an x, y coordinate system for calculation purposes as shown where, for example, point F has coordinates $(60,15)$.

a. State the coordinates of the points B, C, and D.

B has coordinates $(0,30)$, C has coordinates $(30,45)$, and D has coordinates $(60,30)$.

b. If curve BCD is a parabola with equation $y = a(x-30)^2 + k$, find the values of a and k.

The maximum turning point of the parabola BCD is at $(30,45)$, therefore $k = 45$ and $y = a(x-30)^2 + 45$

Substitute $(0,30)$ in $y = a(x-30)^2 + 45$

$30 = a(-30)^2 + 45$

$a = -\frac{15}{900} = -\frac{1}{60}$

For the parabola BCD, $a = -\frac{1}{60}$ and $k = 45$

c. State the coordinates of the points A and E.

A has coordinates $(0,15)$, and E has coordinates $(30,30)$.

Question 1

d. If curve AE has equation $y = m\log_e(x+10) + n$ find the values of m and n, correct to two decimal places.

Substitute (0,15) $\quad 15 = m\log_e 10 + n$

$n = 15 - m\log_e 10 \quad$ [1]

Substitute (30,30) $\quad 30 = m\log_e 40 + n \quad$ [2]

Substitute [1] in [2] $\quad 30 = m\log_e 40 + 15 - m\log_e 10$

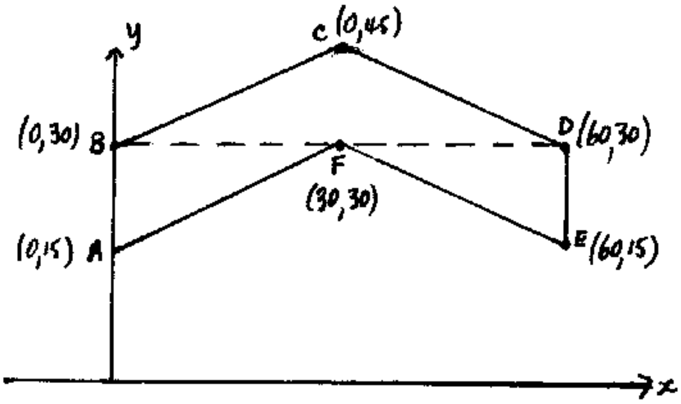
$15 = m\log_e(\frac{40}{10})$

$m = \frac{15}{\log_e 4} = 10.82$

Substitute $m = 10.82$ in [1] $\quad n = 15 - 10.82\log_e 10 = -9.91$

The architect plans to have the span of the bridge painted (as shaded in diagram on page 1). To find the area to be painted the architect approximates the bridge section by a hexagon, $ABCDFE$

e.



Area (ΔABF) = Area (ΔDEF)

$= \frac{1}{2} \times 30 \times 15$

$= 225 \text{ m}^2$

Area (ΔBCD) $= \frac{1}{2} \times 60 \times 15$

$= 450 \text{ m}^2$

Total area $= 450 + 225 + 255$

$= 900 \text{ m}^2$

f. If the area of the region between the arch AEF and the water level OG is 1451 m², find the exact area of the bridge section $ABCDFE$

$$\text{Area required} = \int_0^{60} \left(-\tfrac{1}{60}(x-30)^2 + 45\right) dx - 1451$$

$$= \left[-\tfrac{1}{180}(x-30)^3 + 45x\right]_0^{60} - 1451$$

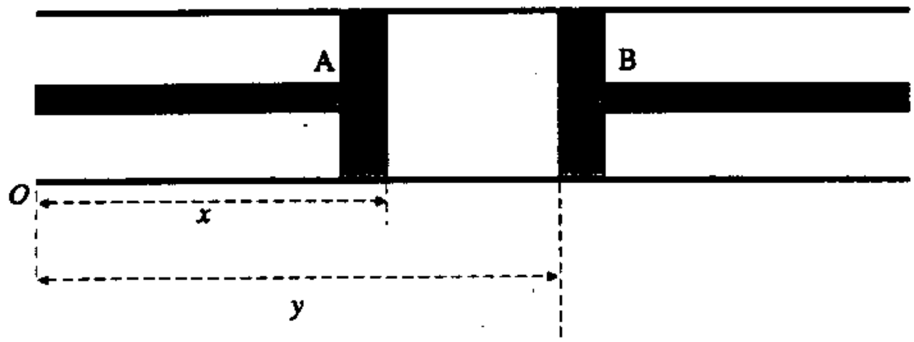
$$= -\tfrac{1}{180}(60-30)^3 + 45(60) + \tfrac{1}{180}(-30)^3 - 1451$$

$$= -150 + 2700 - 150 - 1451$$

$$= 949 \text{ m}^2$$

Question 2

Two pistons A and B move backwards and forwards in a cylinder as shown.



The distance x centimetres of the right hand end of piston A from the point O at time t seconds is modelled by the formula $x = 3\sin(2t) + 3$ and the distance y centimetres of the left hand end of piston B from the point O at time t seconds is modelled by the formula $y = 3\sin(3t - \frac{\pi}{4}) + 8$.
The pistons are set in motion at time $t = 0$.

a. **i.** State the amplitude of the motion of piston A.
Amplitude = 3

ii. Show that the maximum and minimum values of x are 6 and 0 respectively.
The amplitude of motion is 3 and the basic shape is translated 3 units vertically up,
minimum value $= -3 + 3 = 0$ and maximum value $= 3 + 3 = 6$

iii. Show that when $t = \frac{\pi}{4}$, the right hand end of piston A is at its maximum distance from O.
For maximum or minimum distance let $\frac{dx}{dt} = 0$

$6\cos(2t) = 0$

$\cos(2t) = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}$

t	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$
$\frac{dx}{dt}$	+	0	−
	/	−	\

x is maximum when $t = \frac{\pi}{4}$

iv. Find the next four t values $t > \frac{\pi}{4}$ for which $x = 6$.
next four values $= \frac{\pi}{4} + \pi,\ \frac{\pi}{4} + 2\pi,\ \frac{\pi}{4} + 3\pi,\ \frac{\pi}{4} + 4\pi$

$= \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}$

Question 2

b. i. The maximum value of y is 10. State the minimum value of y.

Amplitude is 2 and the basic shape is translated 8 units vertically up,

minimum value $= -2 + 8 = 6$

ii. Show that y attains its minimum value when $t = \frac{7\pi}{12}$.

If $y = 6$, then $6 = 2\sin(3t - \frac{\pi}{4}) + 8$

$$\sin(3t - \tfrac{\pi}{4}) = -1$$

$$3t - \tfrac{\pi}{4} = \tfrac{3\pi}{2}$$

$$3t = \tfrac{7\pi}{4}$$

$$t = \tfrac{7\pi}{12}$$

iii. Find all other values of t, , for which y attains its minimum value.

For the other values of t when y is minimum,

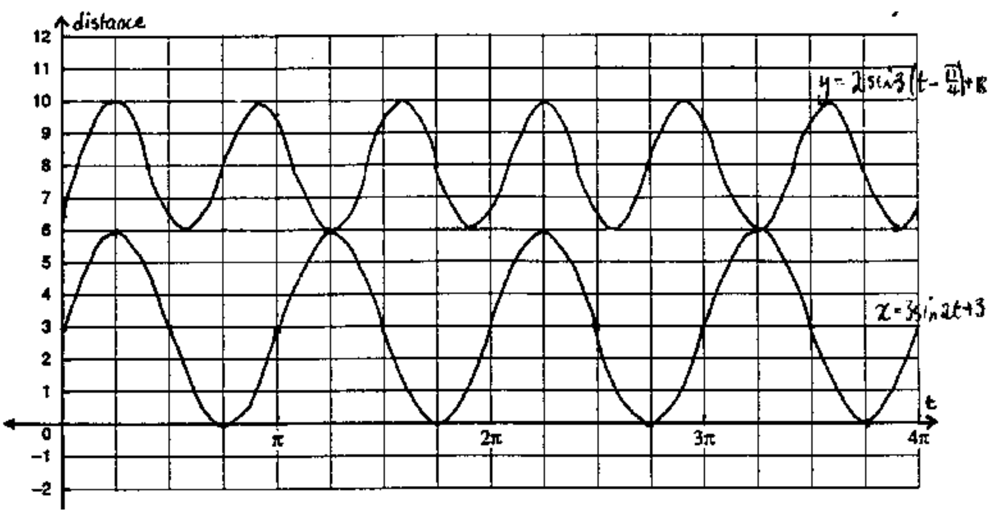
$$3t - \tfrac{\pi}{4} = \tfrac{7\pi}{2}, \tfrac{11\pi}{2}, \tfrac{15\pi}{2}, \tfrac{19\pi}{2}, \tfrac{23\pi}{2}$$

$$3t = \tfrac{15\pi}{4}, \tfrac{23\pi}{4}, \tfrac{31\pi}{4}, \tfrac{39\pi}{4}, \tfrac{47\pi}{4}$$

$$t = \tfrac{5\pi}{4}, \tfrac{23\pi}{12}, \tfrac{31\pi}{12}, \tfrac{13\pi}{4}, \tfrac{47\pi}{12}$$

c. On the one set of axes below, draw graphs of $x(t)$ and $y(t)$.

equation	period	amplitude	max. value	min.value	vert. intercept
$x = 3\sin 2t + 3$	$\frac{2\pi}{2} = \pi$	3	6	0	$3\sin 0 + 3 = 3$
$y = 2\sin 3(t - \frac{\pi}{12}) + 8$	$\frac{2\pi}{3}$	2	10	6	$2\sin(-\frac{\pi}{4}) + 8 \approx 6.6$



Question 2

d. **i.** State the time when the pistons first touch each other.

The pistons first touch each other when $t = \frac{5\pi}{4}$ seconds.

ii. How many seconds are there between the first and second times the pistons touch?

The pistons touch each other a second time when $t = \frac{13\pi}{4}$ seconds.

Time between first and second time the pistons touch $= \frac{13\pi}{4} - \frac{5\pi}{4} = \frac{8\pi}{4} = 2\pi$ seconds.

e. Let T_n seconds be the time at which the pistons meet for the *n*th time.
Then $T_n = a\pi + bn\pi$, where a and b are constants. Find the values of a and b.

When $n = 1$, $T_1 = \frac{5\pi}{4}$

$$\frac{5\pi}{4} = a\pi + b\pi$$
$$\frac{5}{4} = a + b$$
$$5 = 4a + 4b \qquad [1]$$

When $n = 2$, $T_2 = \frac{13\pi}{4}$

$$\frac{13\pi}{4} = a\pi + 2b\pi$$
$$\frac{13}{4} = a + 2b$$
$$13 = 4a + 8b \qquad [2]$$

[2] − [1] $\quad 8 = 4b, \quad b = 2$

sub $b = 2$ in [1] $\quad 5 = 4a + 8, \quad a = -\frac{3}{4}$

f. At what time is the right hand end of piston A first 4 centimetres from *0*?
Give your answer to the nearest one-hundredth of a second.

Let $x = 4$, $\quad 4 = 3\sin 2t + 3$

$$\sin 2t = \frac{1}{3}$$
$$2t = 0.3398$$
$$t = 0.17 \text{ seconds}$$

The right hand end of piston A is first 4 centimetres from *0* after 0.17 seconds.

Question 2

g. i. Find the average rate of change of the position with respect to time (average speed) of piston B in the first 0.2 seconds.

When $t = 0$, $y = 2\sin(-\frac{\pi}{4}) + 8 = 6.5858$

When $t = 0.2$, $y = 2\sin\left(3(0.2) - \frac{\pi}{4}\right) + 8 = 7.6313$

Average rate of change = $\frac{7.6313 - 6.5858}{0.2 - 0} = 5.23$ cm/sec

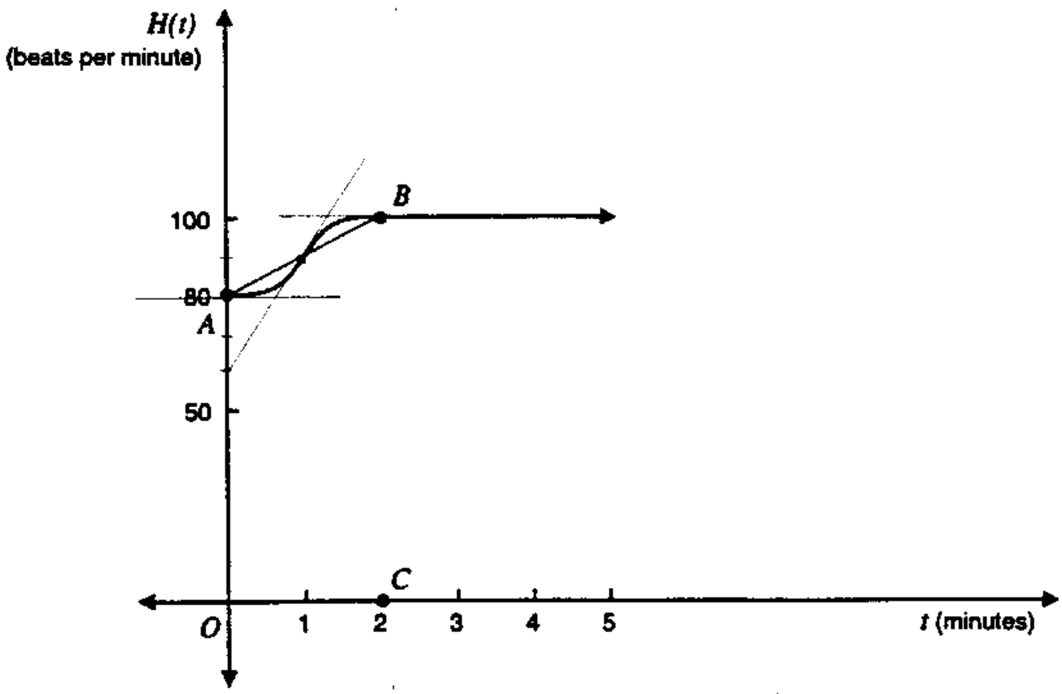
ii. Use calculus to find the speed of piston B at time $t = 0.2$.

Speed $= \frac{dy}{dt} = 6\cos(3t - \frac{\pi}{4})$

When $t = 0.2$, $\frac{dy}{dt} = 6\cos(0.6 - \frac{\pi}{4}) = 5.90$ cm/sec

Question 3

During a mild five minute exercise sequence a student's heartrate $H(t)$, in beats per minute, at time t minutes, is monitored and graphed as shown below. The student's heart pumps a volume of 70 millilitres of blood with each heart beat.



Question 3

a. From the graph estimate

i. the student's heartrate when $t = 0$ and when $t = 1$ and when $t = 2$.

When $t = 0$, $H(0) = 80$ beats per minute

When $t = 1$, $H(1) = 90$ beats per minute

When $t = 2$, $H(2) = 100$ beats per minute

ii. the student's rate of change of heartrate (in beats per minute) when $t = 0$ and when $t = 1$ and when $t = 2$.

Rate of change of heartrate can be estimated by the gradient of the tangent to the curve. The required tangents have been sketched on the previous graph.

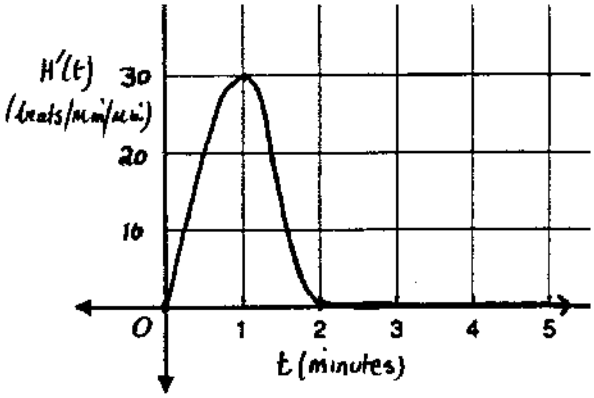
When $t = 0$, rate of change of heartrate = 0 beats per minute per minute.

Assuming that the tangent passes throught the points $(0,60)$ and $(1,90)$, then when

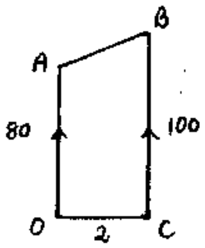
$t = 1$, rate of change of heartrate = $\frac{90-60}{1-0} = 30$ beats per minute per minute.

When $t = 2$, rate of change of heartrate = 0 beats per minute per minute.

b. On the set of axes below, sketch the graph of the rate of change of heartrate, $H'(t)$, over the five-minute exercise program, taking care to b]put a scale on the vertical axis.



c. On the original graph, on page 6, draw the straight line segment from A to B.
Calculate the area of the trapezium $OABC$, and hence estimate the volume of blood pumped by the student's heart in the first two minutes of the exercise program.



Area $(OABC) = \frac{2}{2}(80 + 100)$

$= 180$ beats

Volume of blood pumped over the first 2 minutes

$= 180 \times 70 = 12600$ ml

Question 3

The function $f: R \to R$ where $f(t) = at^3 + bt^2 + ct + d$ has a local minimum at $(0,80)$ and a local maximum at $(2,100)$. For parts **d.**, **e.** and **f.** below, assume that the student's heartrate **for the first two minutes** is modelled by the function *f*.

d. **i.** Show that $c = 0$ and $d = 80$.

$f(0) = 80, \quad 80 = 0 + 0 + 0 + d$

$d = 80$

$f(t) = at^3 + bt^2 + ct + 80$ and $f'(t) = 3at^2 + 2bt + c$

$f'(0) = 0, \quad 0 = 0 + 0 + c$

$c = 0$

ii. Find the values of *a* and *b*.

$f(t) = at^3 + bt^2 + 80$

sub (2,100) $\quad 100 = a(2)^3 + b(2)^2 + 80$

$20 = 8a + 4b$

$5 = 2a + b \qquad [1]$

$f'(t) = 3at^2 + 2bt$

$f'(2) = 0, \quad 0 = 3a(2)^2 + 2b(2)$

$0 = 12a + 4b$

$0 = 3a + b \qquad [2]$

[2]-[1] $\quad -5 = a$

sub $a = -5$ in [1] $\quad 5 = -10 + b$

$b = 15$

$a = -5$ and $b = 15$

e. **i.** Write down a rule for the rate of change of heartrate, $f'(t)$, beats per minute per minute, for $0 \le t \le 2$.

$f(t) = -5t^3 + 15t^2 + 80 \qquad 0 \le t \le 2$

$f'(t) = -15t^2 + 30t \qquad 0 \le t \le 2$

ii. Find when the heartrate is increasing most quickly.

Heartrate is increasing most quickly when $f'(t)$ is maximum.

If $f''(t) = 0$ then $-30t + 30 = 0$

$t = 1$

t	0.5	1	1.5
$f''(t)$	+	0	−
	/	−	\

Heartrate is increasing most quickly after 1 minute.

Question 3

f. Write down and find a definite integral which gives the total number of heartbeats during the first two minutes, and hence calculate the volume of blood pumped during this time.

Total number of heartbeats during the first two minutes $= \int_0^2 (-5t^3 + 15t^2 + 80)\, dt$

$$= \left[-\tfrac{5}{4}t^4 + 5t^3 + 80t\right]_0^2$$

$$= -\tfrac{5}{4}(2)^4 + 5(2)^3 + 80(2)$$

$$= 180$$

Volume of blood pumped during this time $= 180 \times 70 = 12600$ ml

Question 4

Victoria Jones is a contestant in the long-jump event at the world championships. In a particular jump, Victoria jumps X metres. X is a normally distributed random variable with mean 7.2 metres and standard deviation 0.2.

a. For any jump, find the probability that Victoria jumps

i. more than 7.5 metres.

$\Pr(X > 7.5) = \Pr(Z > \frac{7.5-7.2}{0.2}) = \Pr(Z > 1.5) = 1 - \Pr(Z < 1.5)$

$= 1 - 0.9332 = 0.0668$

ii. less than 7.0 metres.

$\Pr(X < 7.0) = \Pr(z < \frac{7.0-7.2}{0.2}) = \Pr(Z < -1) = \Pr(Z > 1) = 1 - \Pr(Z < 1)$

$= 1 - 0.8413 = 0.1587$

iii. between 7.0 and 7.5 metres.

$\Pr(7.0 < X < 7.5) = \Pr(X < 7.5) - \Pr(X < 7.0) = \Pr(z < 1.5) - \Pr(Z < -1)$

$= 0.9332 - 0.1587$

$= 0.7745$

Question 4

During the championships each competitor in the long jump event has five jumps.

b. In her first five jumps, find the probability, stated to three significant figures, that

i. Victoria's first three jumps are all less than 7.5 metres and both her last two jumps are more than 7.5 metres.

Pr(first three jumps are less than 7.5m and the last two jumps are more than 7.5m)

$= (0.9332)^3(0.0668)^2$

$= 0.00363$

ii. at least three of Victoria's jumps are more than 7.5 metres.

Let J denote the number of jumps more than 7.5 metres; J Bi(5.0.0668)

Pr(at least three jumps more than 7.5 metres)

$= \Pr(J = 3) + \Pr(J = 4) + \Pr(J = 5)$

$= {}^5C_3(0.0668)^3(0.9332)^2 + {}^5C_4(0.0668)^4(0.9332) + {}^5C_5(0.0668)^5(0.9332)^0$

$= 0.00269$

iii. Victoria's final jump is more than 7.5 metes, given that at least three of her five jumps are more than 7.5 metres.

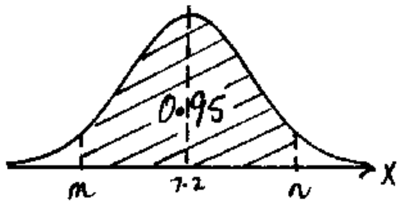
Pr(last jump is more than 7.5 metres \ at least three jumps are more than 7.5 metres)

$= \dfrac{\text{Pr(at least three jumps included the last jump are more than 7.5 metres)}}{\text{Pr(at least three jumps are more than 7.5 metres)}}$

$= \dfrac{{}^4C_2(0.0668)^2(0.9332)^2 \times 0.0668}{0.00269}$

$= 0.579$

c. Between what two distances, symmetrically placed about the mean, would 95% of Victoria's jumps be expected to lie?



$\Pr(m < X < n) = 0.95$

$\Pr(7.2 < X < n) = 0.475$

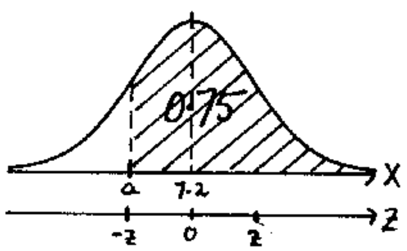
$\Pr(X < n) = 0.475 + 0.5 = 0.975$

$\frac{n-7.2}{0.2} = 1.96$ and $\frac{m-7.2}{0.2} = -1.96$

$n = 7.59$ and $\text{m} = 6.81$

95% of Victoria's jumps should lie between 6.81 metres and 7.59 metres.

d. If 75 per cent of Victoria's jumps are greater than a metres, what is the value of a?



$\Pr(X > a) = 0.75$

$\Pr(Z > -z) = 0.75$

$\Pr(Z < z) = 0.75$

$z = 0.674$

$\frac{a-7.2}{0.2} = -0.674$

$a = 7.07$ metres (to 2 decimal places)

Question 4

e. During training for the championships, Victoria had sixty practice jumps. How many of these jumps would be expected to be more than 7.5 metres?

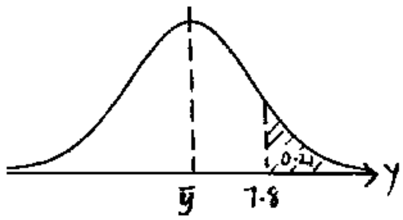
J Bi(60,0.0668)

$E(J) = 60 \times 0.0668 = 4.008$

Victoria can expect four of these practice jumps to be more than 7.5 metres.

After her five world championship jumps, Victoria's best jump was 7.8 metres. She was leading the event with Valda Bos's final jump still to come. In a particular jump, Valda jumps Y metres; where Y is a normally distributed random variable with variance 0.16.

f. If the probability that Valda's final jump is more than 7.8 metres is 0.21, find the mean of the distances that Valda Bos jumps, correct to two decimal places.



$$\Pr(Y > 7.8) = 0.21$$

$$\Pr\left(Z > \frac{7.8-\bar{y}}{\sqrt{0.16}}\right) = 0.21$$

$$\Pr\left(Z < \frac{7.8-\bar{y}}{\sqrt{0.16}}\right) = 0.79$$

$$\frac{7.8-\bar{y}}{0.4} = 0.803$$

$$7.8 - \bar{y} = 0.3212$$

$$\bar{y} = 7.48 \text{ metres}$$

The mean distance Valda jumps is 7.48 metres.

END OF SUGGESTED SOLUTIONS

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CHEMISTRY ASSOCIATES
P.O. BOX 2227
KEW, VIC., 3101
AUSTRALIA
TEL:(03) 9817 5374 FAX: (03) 9817 4334