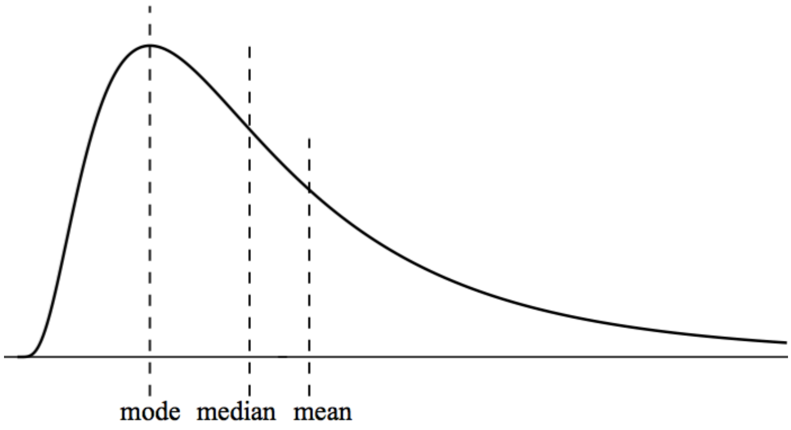
FURTHER MATHEMATICS – UNITS 3&4
2018
Written examination 1 Solutions

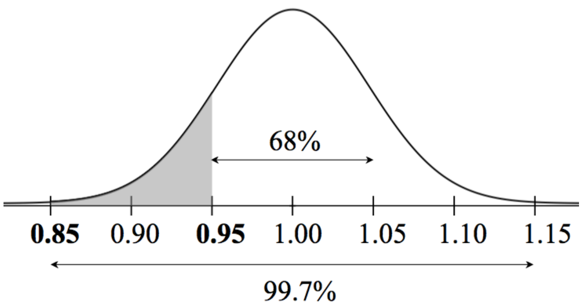
SECTION A – Core
Data analysis

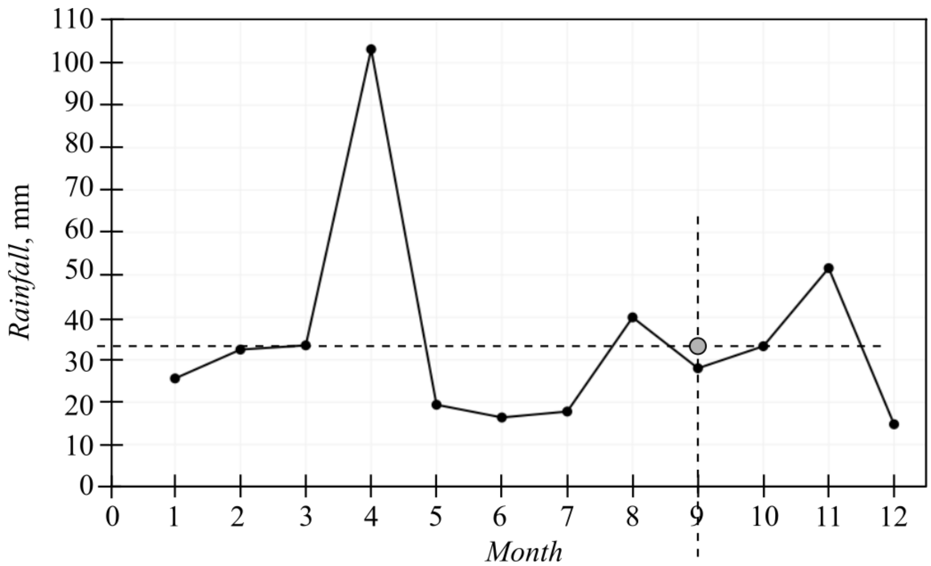
<table>
<tr><th>Question</th><th>Answer</th><th>Solution</th></tr>
<tr><td>1</td><td>D</td><td>83.1 84 84.5 85.6 87 89 90.8 90.9 90.9 93.2 94 94.6 96.4 99.5 103.4 104 104.3 105.7
The median is between 90.9 and 93.2 $\Rightarrow \frac{90.9+93.2}{2}=92.05$
Minimum = 83.1 $Q_1 = 87$ Median = 92.05 $Q_3 = 99.5$ Maximum = 105.7</td></tr>
<tr><td>2</td><td>A</td><td>$IQR = Q_3 - Q_1$
$= 99.5 - 87$
$= 12.5$

Lower bound $= Q_1 - 1.5 \times IQR$
$= 87 - 1.5 \times 12.5$
$= 68.25 \Rightarrow$ no outliers on this side

Upper bound $= Q_3 + 1.5 \times IQR$
$= 99.5 + 1.5 \times 12.5$
$= 118.25 \Rightarrow$ no outliers on this side</td></tr>
<tr><td>3</td><td>E</td><td>mode < median < mean $\Rightarrow$ Positively skewed.
</td></tr>
<tr><td>4</td><td>A</td><td>Both variables are categorical ordinal variables.
Age: under 30 and over 30 are in order.
Driving over the speed limit: never, rarely, sometimes are in order.</td></tr>
<tr><td>5</td><td>E</td><td>Option A … true
$\frac{13}{240} \times 100 \approx 5.4\%$ of the people surveyed sometimes drive over the speed limit

Option B … true
240 people participated in this survey.</td></tr>
</table>

		Option C … true $\frac{154}{240}\times 100 \approx 64\%$ of all the people surveyed **never** *drive over the speed limit*. **Option D** … true 100 people **under 30** participated in this survey. **Option E** … **not** true People **over 30** who **rarely** *drive over the speed limit*: $\frac{39}{140}\times 100 \approx 28\%$
6	**B**	A random scatterplot of the residuals indicates a **linear** model and **not** a non-linear model. The response variable is the variable on the **vertical** axis and **not** the one on the horizontal axis.
7	**D**	Less than 3 books for girls = 25% (first quartile) Less than 3 books for boys = 50% (median) 25% of 120 = 30 girls 50% of x boys = 30 boys Total number of boys = $\frac{30}{0.5}$ = 60 boys were surveyed
8	**C**	 99.7% – 68% = 31.7% $\frac{31.7}{2} = 15.85\%$
9	**B**	$z = \frac{x-\mu}{\sigma}$ $-1.6 = \frac{x-1}{0.05} \Rightarrow x - 1 = 0.05 \times (-1.6)$ $x = 1 - 0.08$ $x = 0.92$ cm
10	**A**	$L = 0.94 \times R + 1.06$
11	**C**	$b = r\frac{s_y}{s_x}$ $= 0.72 \times \frac{12.8}{7.9}$ $= 1.17$
12	**E**	x^2 will **not** linearise this data.

13	**A**	seasonal index $= \dfrac{\text{actual revenue}}{\text{deseasonalised revenue}}$ Monday has a seasonal index of 0.68 $\Rightarrow 0.68 = \dfrac{\text{actual revenue}}{\text{deseasonalised revenue}}$ $\text{deseasonalised revenue}_{\text{Monday}} = \dfrac{\text{actual revenue}}{0.68}$ $\approx 1.47 \times \text{actual revenue}$ The actual revenue must be increased by 47%. Thursday has a seasonal index of 1.35 $\Rightarrow 1.35 = \dfrac{\text{actual revenue}}{\text{deseasonalised revenue}}$ $\text{deseasonalised revenue}_{\text{Thursday}} = \dfrac{\text{actual revenue}}{1.35}$ $\approx 0.74 \times \text{actual revenue}$ The actual revenue must be decreased by 26%.
14	**D**	$\text{deseasonalised revenue}_{\text{Thursday}} = \dfrac{\text{actual revenue}}{1.35}$ $= \dfrac{1250}{1.35}$ $= 925.926$ $\approx \$926$
15	**A**	An irregular pattern.
16	**C**	 33 mm

Recursion and financial modelling

Question	Answer	Solution
17	**A**	This is a simple interest account because a flat rate of \$28 is added at the end of each year. r% of \$800 = \$28 $r = \frac{28}{800} \times 100$ $= 3.5\%$
18	**B**	The value of the asset at the end of year $n + 1$ is \$4000 less than the value of the asset at the end of year n. $V_{n+1} = V_n - 4000$, where $V_0 = \$50000$
19	**A**	Geometric decay = 100% – 30% = 70% = 0.7 The height of the hedge at the end of the 1st year = $0.7 \times (0.5 + 0.4)$ $= 0.7 \times 0.9$ The height of the hedge at the end of the 2nd year = $0.7 \times (0.7 \times 0.9 + 0.4)$ $= 0.7^2 \times 0.9 + 0.7 \times 0.4$ The height of the hedge at the end of the 3rd year = $0.7 \times (0.7^2 \times 0.9 + 0.7 \times 0.4 + 0.4)$ $= 0.7^3 \times 0.9 + 0.7^2 \times 0.4 + 0.7 \times 0.4$
20	**D**	Loan A: Using available CAS technology, N: 52 I(%): 5 PV: -15000 Pmt: 0 **FV: 15768.688** ⇒ interest = 15768.688 – 15000 = 768.688 ≈ \$769 PpY: 52 CpY: 52 Loan B: Using available CAS technology, N: 1 I(%): 8 PV: -15000 Pmt: 0 **FV: 16200** ⇒ interest = 16200 – 15000 = \$1200 PpY: 1 CpY: 1
21	**C**	Using Finance solver on available CAS technology, the future value of the loan at the end of the 2nd month is N: 2 I(%): 3.6 PV: 460000 Pmt: –2500 **FV: -457756.64** PpY: 12 CpY: 12

		Using Finance solver on available CAS technology, the future value of the loan at the end of the 3rd month is N: 3 I(%): 3.6 PV: 460000 Pmt: –2500 **FV: -456629.9** PpY: 12 CpY: 12 Loan reduction $= 457756.64 - 456629.9 \approx \$1127$
22	**D**	N: 4 **I(%): 5.33333** PV: −150000 Pmt: 2000 FV: 150000 PpY: 4 CpY: 4
23	**E**	Amount per month $= \dfrac{21600}{12} = \$1800$ N: 36 I(%): 6.4 **PV: −219999.9986 ≈ 220000** Pmt: 1800 FV: 195201.33 PpY: 12 CpY: 12
24	**D**	**Option A** … true **N: 36.02389 ≈ 3 years** I(%): 3.8 PV: 120000 Pmt: −3530 FV: 0 PpY: 12 CpY: 12 **Option B** … true N: 36 I(%): 3.8 PV: 120000 Pmt: −2500 **FV: −39294.76** PpY: 12 CpY: 12 **Option C** … true **N: 47.98 ≈ 4 years** I(%): 3.8 PV: 120000 Pmt: −2700 FV: 0 PpY: 12 CpY: 12

		Option D … false A higher monthly payment will decrease the life of the loan and **not increase**. **Option E** … true N: 1 I(%): 3.8 PV: 120000 Pmt: –3530 **FV: –116850** ⇒ 120000 – 116850 = $3150 PpY: 12 CpY: 12

SECTION B – Applications
Module 1 – Matrices

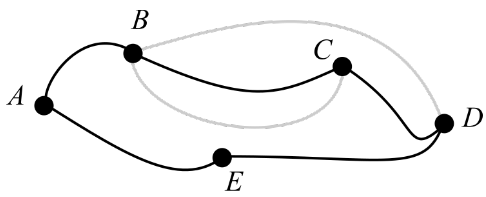
Question	Answer	Solution
1	C	$mA + nB = I$ $m\begin{bmatrix} -1 & 4 \\ 2 & -1 \end{bmatrix} + n\begin{bmatrix} -2 & 6 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -m & 4m \\ 2m & -m \end{bmatrix} + \begin{bmatrix} -2n & 6n \\ 3n & -2n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -m-2n & 4m+6n \\ 2m+3n & -m-2n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ … equate corresponding elements $-m - 2n = 1$ … [1] $4m + 6n = 0$ … [2] Using available CAS technology solve the system of simultaneous equations [1] and [2]. $m = 3$ and $n = -2$. **Alternative ‘by hand’ method** Multiply [1] by 4. $-4m - 8n = 4$ … [3] $4m + 6n = 0$ … [2] Add [2] and [3]. $-2n = 4$ … divide the equation by 2 $n = -2$ Substitute $n = -2$ into [1] $-m - 2 \times (-2) = 1$ … simplify $-m + 4 = 1$ subtract 4 from both sides of the equation $-m = -3$ … multiply the equation by (–1) $m = 3$
2	E	$a_{23} = 2 - 2 \times 3$ $= -4$ $b_{23} = 2 \times 2 + 3$ $= 7$ $2a_{23} - 3b_{23} = 2 \times (-4) + 3 \times 7$ $= 13$
3	E	$(C + D)(C - D) = C \times C - C \times D + D \times C - D \times D$ $= C^2 - CD + DC - D^2$

4	**A**	$\begin{bmatrix} 28 & 35 \\ 37 & 42 \end{bmatrix} \cdot \begin{bmatrix} 120 \\ 75 \end{bmatrix} = \begin{bmatrix} 28\times120+35\times75 \\ 37\times120+42\times75 \end{bmatrix} \begin{matrix} \text{July} \\ \text{August} \end{matrix}$
5	**C**	Using available CAS technology solve each system of linear equations or calculate the determinant of each square matrix. $\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ has a unique solution ⇒ **consistent independent** $\begin{bmatrix} 6 & 2 \\ -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$ has infinitely many solutions ⇒ consistent dependent $\begin{bmatrix} -1 & 5 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ has a unique solution ⇒ **consistent independent** $\begin{bmatrix} 7 & 2 \\ -14 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ has no solution ⇒ inconsistent
6	**C**	Matrix P **cannot** be used to change $\begin{bmatrix} S \\ T \\ E \\ A \\ M \end{bmatrix}$ to $\begin{bmatrix} M \\ E \\ A \\ T \\ S \end{bmatrix}$.
7	**D**	Substitute the matrices given into $M_{n+1} = TM_n + A$. $M_1 = \begin{bmatrix} 0.24 & 0.15 & 0 \\ 0 & 0.36 & 0.72 \\ 0.76 & 0.49 & 0.28 \end{bmatrix} \cdot \begin{bmatrix} 1200 \\ 850 \\ 950 \end{bmatrix} + \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix} = \begin{bmatrix} 715.5 \\ 1390 \\ 294.5 \end{bmatrix}$ $M_2 = \begin{bmatrix} 0.24 & 0.15 & 0 \\ 0 & 0.36 & 0.72 \\ 0.76 & 0.49 & 0.28 \end{bmatrix} \cdot \begin{bmatrix} 715.5 \\ 1390 \\ 294.5 \end{bmatrix} + \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix} = \begin{bmatrix} 680.22 \\ 2408.44 \\ 2311.34 \end{bmatrix}$ $M_3 = \begin{bmatrix} 0.24 & 0.15 & 0 \\ 0 & 0.36 & 0.72 \\ 0.76 & 0.49 & 0.28 \end{bmatrix} \cdot \begin{bmatrix} 680.22 \\ 2408.44 \\ 2311.34 \end{bmatrix} + \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix} = \begin{bmatrix} 824.52 \\ 2931.2 \\ 2844.28 \end{bmatrix}$
8	**B**	*loss* $\begin{matrix} & & A & B & C & D \\ & A & 0 & 1 & 0 & 1 \\ win & B & 0 & 0 & 1 & 0 \\ & C & 1 & 0 & 0 & 1 \\ & D & 0 & 1 & 0 & 0 \end{matrix}$

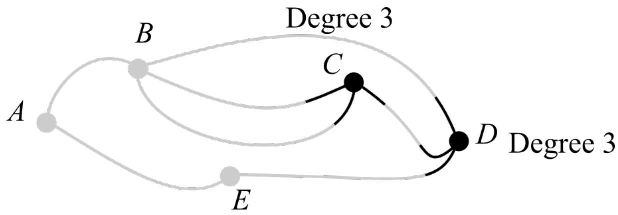
Module 2 – Networks and decision mathematics

<table>
<tr><th>Question</th><th>Answer</th><th>Solution</th></tr>
<tr><td>1</td><td>E</td><td>The graph is connected, has three vertices of even degrees and two vertices of odd degrees, has four faces and seven edges, is planar but does not have any isolated vertices.</td></tr>
<tr><td>2</td><td>A</td><td>

Option A
There is at least one hamiltonian cycle shown below.



There is no eulerian circuit because the graph has vertices of odd degree.



Option B
No possible hamiltonian cycles; however, there is at least one eulerian circuit.
One possible eulerian circuit:
A-*B*(top edge)-*C*(top edge)-*D*(bottom edge)-*C*(bottom edge)-*B*-*E*-*A*

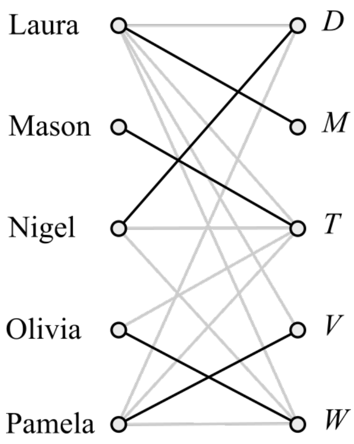
Option C
No possible hamiltonian cycles. No eulerian circuits because of the vertex of degree 1 (vertex *D*).

Option D
The graph has at least one hamiltonian cycle (*A*-*B*-*C*-*D*-*E*-*A*) and at least one eulerian circuit (*B*-*A*-*E*(bottom edge)-*D*(top edge)-*B*-*C*-*D*-*B*).

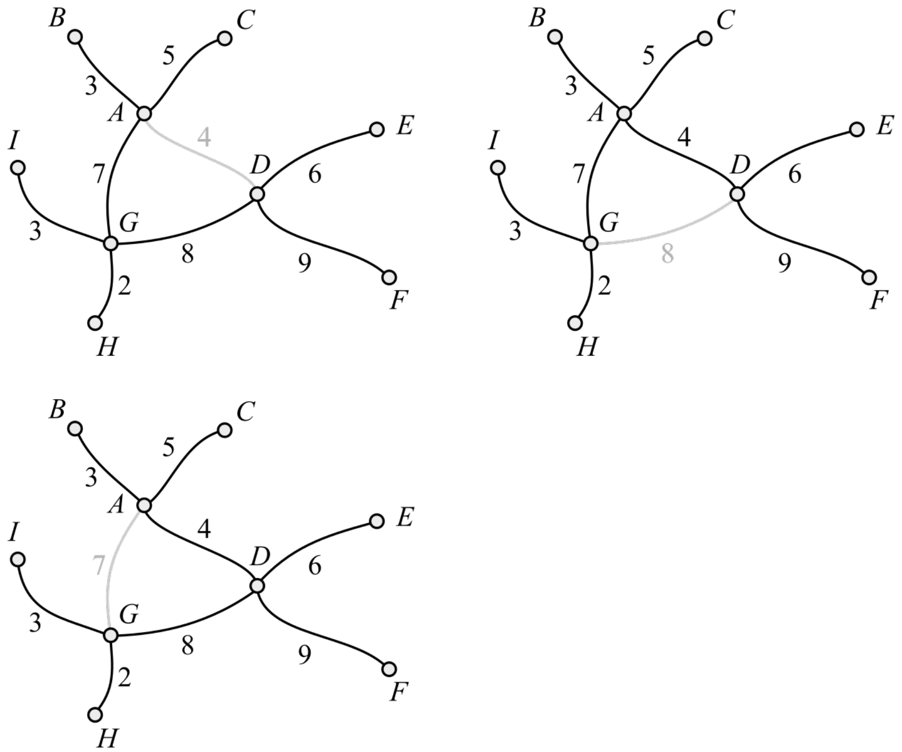
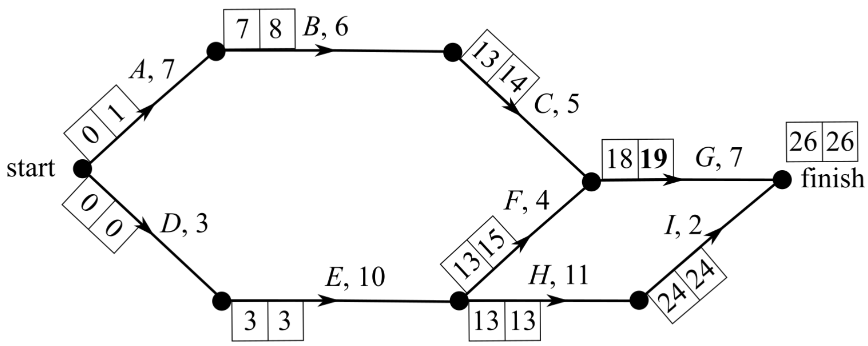
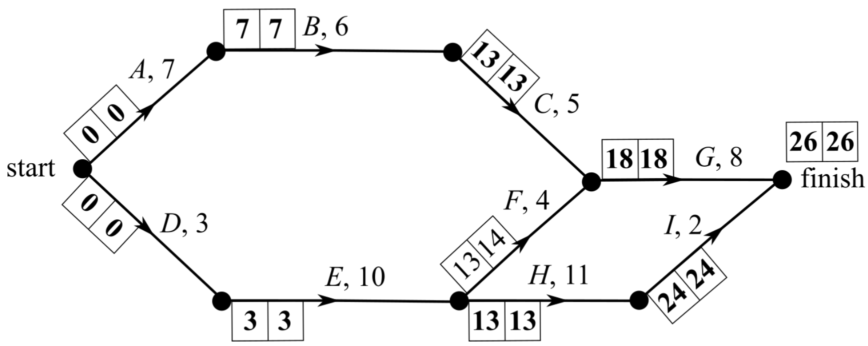
Option E
Since *D* is an isolated vertex no cycles or circuits are possible.

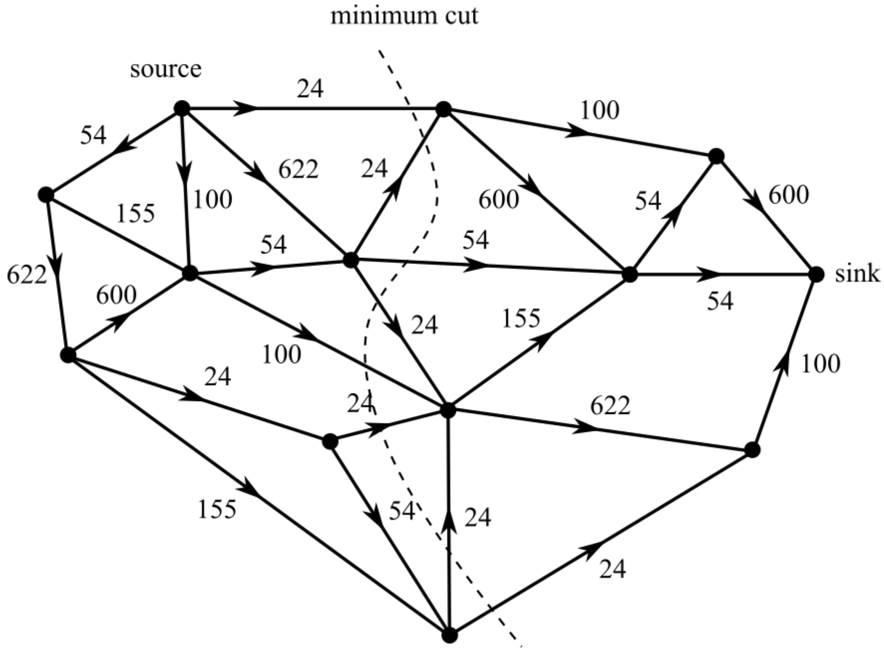
</td></tr>
<tr><td>3</td><td>C</td><td>

Nigel is going to wash the dog.



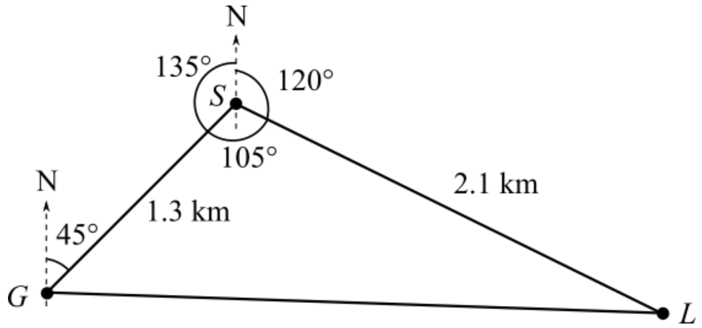
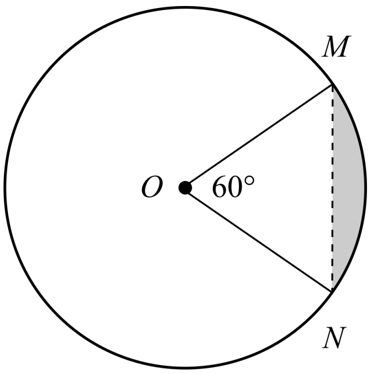
</td></tr>
</table>

<table>
<tr><td>4</td><td>C</td><td>The 3 possible spanning trees are shown below.
</td></tr>
<tr><td>5</td><td>A</td><td>The minimum spanning tree (without edge DG) = 3 + 5 + 4 + 6 + 9 + 7 + 3 + 2
= 39</td></tr>
<tr><td>6</td><td>D</td><td>
LST for activity G is 19 minutes.</td></tr>
<tr><td>7</td><td>B</td><td>
Changing G, 7 to G, 8 produces the times shown below.
Second critical path: A-B-C-G.</td></tr>
</table>

8	**E**	The minimum cut crosses 8 edges $n = 8$ $m = 24 + 24 + 54 + 24 + 100 + 24 + 24 + 24$ $= 298$ megabites 

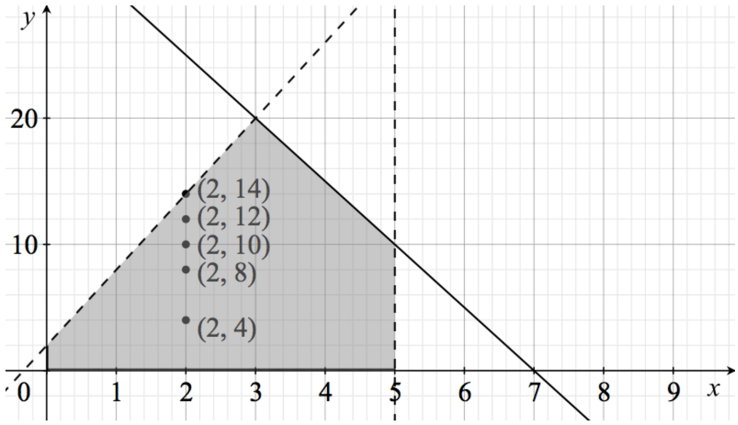
Module 3 – Geometry and measurement

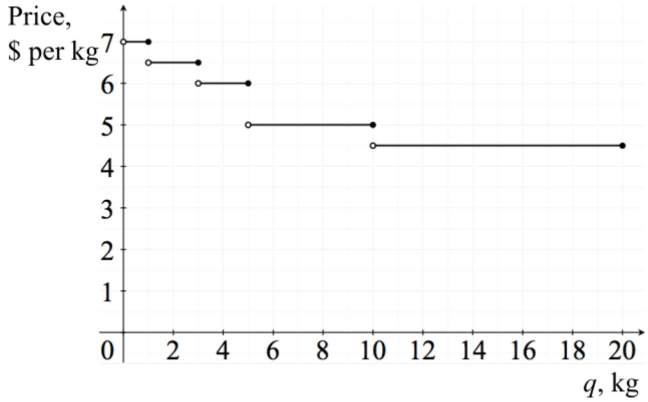
Question	Answer	Solution
1	**A**	Length of smaller cube $= 5.7 \div 3$ $= 1.9$ cm
2	**D**	Since the device takes the space of one small cube, the volume of this space is $\frac{1}{27}V$.
3	**E**	Using Pythagoras theorem, $s^2 = DH^2 + \left(\frac{AB}{2}\right)^2$ $s = \sqrt{29.6^2 + 7.4^2}$ $= 30.51$ cm
4	**E**	*ABCD* is a regular triangular pyramid which means that the triangular base is an equilateral triangle and the three triangular faces are identical isosceles triangles. Area of the triangular base $= \frac{1}{2} \times 14.8 \times 14.8 \times \sin(60°)$ $= 94.85$ cm^2 $3 \times \text{Area}_{\Delta ABD} = 3 \times \frac{1}{2} \times 14.8 \times 29.6$ $= 657.12$ cm^2 Total surface area $= 94.85 + 657.12$ $= 751.97$ cm^2

5	**A**	Location *A* is on a 60°W longitude and location *B* is on a 30°E longitude. Angular distance = 30° –(–60°) = 90° Time difference = $\frac{90°}{15°}$ = 6 hours
6	**D**	The angular distance between locations *A* and *D* = 54° –(–46°) = 100° The length of arc *AD* = $\frac{100°}{360°} \times 2\pi \times 6400$ = 11170 km
7	**C**	 $\angle GSL$ = 360° – (135° + 120°) = 105° $GL = \sqrt{GS^2 + SL^2 - 2GS \times SL \times \cos(\angle GSL)}$ $= \sqrt{1.3^2 + 2.1^2 - 2 \times 1.3 \times 2.1 \times \cos(105°)}$ = 2.74 km
8	**B**	 Area of sector *OMN* = $\frac{60°}{360°} \times \pi \times 95^2 = \frac{95^2\pi}{6}$ Area of triangle *OMN* = $\frac{1}{2} \times 95^2 \sin(60°)$ Shaded area$_1$ = Area of sector *OMN* – Area of triangle *OMN* $= \frac{95^2\pi}{6} - \frac{1}{2} \times 95^2\sin(60°)$ … take out 95^2 as common factor $= 95^2\left(\frac{\pi}{6} - \frac{\sin(60°)}{2}\right)$ … $95^2 = 9025$ Since the two circles are equal, then the total shaded area = 2 × Shaded area$_1$ Total shaded area = $9025\left(\frac{\pi}{3} - \sin(60°)\right)$

Module 4 – Graphs and relations

Question	Answer	Solution
1	A	Using available CAS technology make y the subject in each equation. $y = -\frac{2}{3}x + \frac{7}{3}$, $y = 6.5x + 3.25$, $y = -2x + \frac{1}{4}$, $y = 9x + 2$, $y = \frac{1}{12}x - \frac{1}{8}$
2	C	(2, 1) and (3, a) lie on the graph of $y = kx^{-2}$. Substitute (2, 1) into $y = kx^{-2}$. $1 = \frac{k}{2^2} \Rightarrow k = 4$ Substitute (3, a) into $y = 4x^{-2}$. $a = \frac{4}{9}$
3	E	Brianna has 60 coins in her moneybox with a total value of \$19.50. $f + t = 60$ … [1] $0.50f + 0.20t = 19.50$ … [2] Using available CAS technology solve the simultaneous linear equations [1] and [2]. **Alternative 'by hand' method** Substitute $f = 60 - t$ into [2]. $0.50 \times (60 - t) + 0.20t = 19.50$ … expand brackets $30 - 0.50t + 0.20t = 19.50$ … collect like terms $0.30t = 10.50$ … divide the equation by 0.30 $t = \frac{10.50}{0.30}$ $t = 35$ coins Substitute $t = 35$ into [1]. $f = 60 - 35$ $= 25$ coins
4	A	**Option A** … false The cost function $C(x) = 1000 + 4.50x$ and **not** $C(x) = 1000 - 4.50x$. **Option B** … true The profit function, $P(x) = R(x) - C(x)$ $= 0.02x^2 + 10x - (1000 + 4.50x)$ $= 0.02x^2 + 5.50x - 1000$. **Option C** … true For the break-even point, use available CAS technology to solve the equation $C(x) = R(x)$ $\Rightarrow x = 125$ schoolbags **Option D** … true Since the break-even point is 125 schoolbags, then for $x < 125$ the company incurs a loss. **Option E** … true Substitute $x = 200$ into $P(x)$. $P(200) = 0.02 \times 200^2 + 5.50 \times 200 - 1000$ $= \$900$

5	**E**	$p \neq 14$ 
6	**B**	$F = ax + 3y$ Substitute each corner point into $F(x)$. At the point (0, 0), $F = a \times 0 + 3 \times 0 = 0$ … [1] At the point (0, 2), $F = a \times 0 + 3 \times 2 = 6$ … [2] At the point (5, 0) $\Rightarrow F = a \times 5 + 3 \times 0 = 5a$ … [3] At the point (5, 10) $\Rightarrow F = a \times 5 + 3 \times 10 = 5a + 30$ … [4] At the point (3, 20) $\Rightarrow F = a \times 3 + 3 \times 20 = 3a + 60$ … maximum For $3a + 60$ to be a maximum, it must be greater than all the other values. From [1]: $3a + 60 > 0 \Rightarrow a > -20$ … true since $a > 0$ From [2]: $3a + 60 > 6 \Rightarrow a > -18$ … true since $a > 0$ From [3]: $3a + 60 > 5a \Rightarrow a < 30$ From [4]: $3a + 60 > 5a + 30 \Rightarrow a < 15$ Since $3a + 60$ must be greater than both $5a$ and $5a + 30$, then $a < 15$. $a \in (0, 15)$
7	**B**	The horse walks 120 m in 2 minutes means that the gradient of the line is $120 \div 2 = 60$ $\Rightarrow$ the first branch has equation $d(t) = 60t$. The horse stops for 3 minutes which means that the horse is 120 m from the starting point between $t = 2$ min and $t = 5$ min $\Rightarrow$ the second branch has equation $d(t) = 120$. The horse turns around (negative gradient) and trots 200 m in 1.6 minutes which means that the gradient of the line is $\frac{-200}{1.6} = -125$ $\Rightarrow$ the third branch has equation $d(t) = -125t + c$ To calculate c substitute point (5, 120) into $d(t) = -125t + c$. $120 = -125 \times 5 + c$ $c = 745$ The third equation is $d(t) = -125t + 745$ between $t = 5$ min and $t = 6.6$ min. $d(t) = \begin{cases} 60t & 0 \le t \le 2 \\ 120 & 2 < t \le 5 \\ -125t + 745 & 5 < t \le 6.6 \end{cases}$

8	**D**	Price, $ per kg 7 6 5 4 3 2 1 0 2 4 6 8 10 12 14 16 18 20 q, kg