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FURTHER MATHEMATICS TRIAL EXAMINATION 2 SOLUTIONS 2017

SECTION A - Core

Data analysis

Question 1 (6 marks)

a. The maximum UV index was greater than 10 in $5+2=7$ cities.

(1 mark)

b. The modal interval is 6 - 8.

(1 mark)

c. In $1+3+6=10$ cities the maximum UV index was less than 6.

This represents $\left(\frac{10}{33}\times\frac{100}{1}\right)\% = 30.3030...\%$.

The percentage of cities is 30.3% (rounded to 1 decimal place).

(1 mark)

d. **i.** median (or Q_2) $= 7.6$.

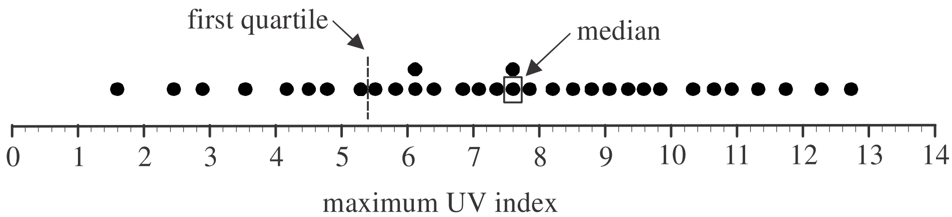
The median is the middle or 17th place of data, shown in the dot plot below.

(1 mark)

ii. first quartile (or Q_1) $= 5.4$

The first quartile is halfway between the 8th and 9th pieces of data as indicated by the dotted line shown below.

(1 mark)



e. The dot plot gives us the exact data values. The histogram only gives us ranges in which the data values lie, for example 6 – 8.

(1 mark)

Question 2 (6 marks)

a. **i.** $\text{range} = 10 - 1.5$

$= 8.5$ **(1 mark)**

Note that we use the piece of data which is an outlier (i.e. 1.5) and **not** the end value of the whisker (i.e. **not** 4).

ii. June – symmetric
October – negatively skewed with an outlier. **(1 mark)**

iii. $\text{lower fence} = Q_1 - 1.5 \times IQR$ (formula sheet)

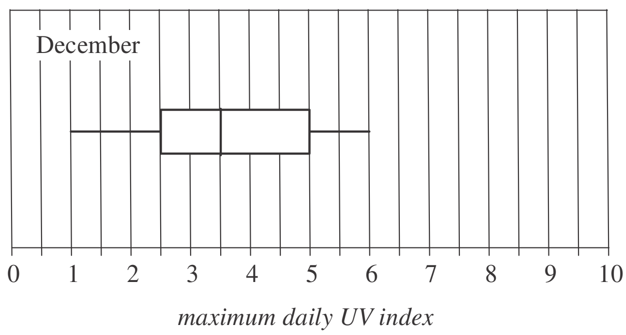
$= 6.5 - 1.5 \times (9.5 - 6.5)$

$= 2$ **(1 mark)**

Since $1.5 < 2$, the *maximum daily UV index* of 1.5 is an outlier. **(1 mark)**

iv. The medians for the months of June and October differ. In June the median *maximum daily UV index* is 10 and in October the median *maximum daily UV index* is 8.5.
Alternatively, an appropriate explanation could involve the IQR's:
The interquartile ranges for the two months differ. In June the interquartile range is 2 and in October the interquartile range is 3. **(1 mark)**

b.



(1 mark)

Question 3 (8 marks)

a. The explanatory variable (which appears on the horizontal axis of the scatterplot) is *latitude*. **(1 mark)**

b. The associations between the *average annual noon UV index* and *latitude* is strong, negative and linear. **(1 mark)**

c. **i.** Enter the two columns of data into your calculator. Remember that the explanatory variable or x variable, is *latitude*. The least squares regression line is

average annual noon UV index $= 14.2969... - 0.19873... \times$ *latitude*

(1 mark) **(1 mark)**

Expressing the intercept and slope values rounded to two decimal places gives *average annual noon UV index* $= 14.30 - 0.20 \times$ *latitude* **(1 mark)**

ii. For every increase of one degree south in *latitude* there is a decrease of 0.20 (to 2 dec. places) in the *average annual noon UV index.* **(1 mark)**

d. **i.** $r^2 = 0.9681$ (coefficient of determination)

$r = \pm\sqrt{0.9681}$

$= \pm 0.9839...$

$= \pm 0.98$ (correct to 2 significant figures)

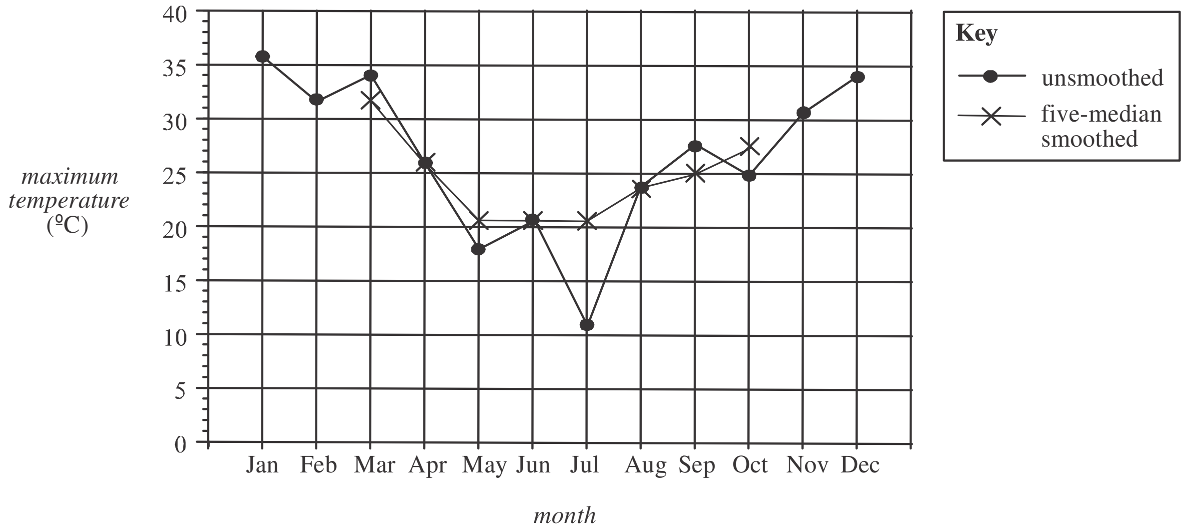
From part **b**. we know that the association is negative; that is, as latitude increases, annual average UV index decreases (which is why the least squares regression line slopes down to the right). So Pearson's correlation coefficient must be negative. So $r = -0.98$. **(1 mark)**

ii. The coefficient of determination tells us that 96.81% of the variation in *average annual noon UV index* can be explained by the variation in *latitude*.

(1 mark)

Question 4 (4 marks)

a.



(1 mark) – two correctly placed crosses
(1 mark) – two more correctly placed crosses

b. From the table given, the monthly seasonal index for the maximum temperature in March is 1.06.

$$\text{seasonal index} = \frac{\text{actual figure}}{\text{deseasonalised figure}} \quad \text{(formula sheet)}$$

$$1.06 = \frac{34}{\text{deseasonalised figure}}$$

$$\text{deseasonalised figure} = \frac{34}{1.06}$$

$$= 32.0754...$$

The deseasonalised value is 32° to the nearest degree.

(1 mark)

c. A monthly seasonal index of 0.85 tells us that the maximum temperatures in July tend to be 15% lower than the monthly average.
You could also say that the July maximum temperatures tend to be 85% of the monthly average.

(1 mark)

Recursion and financial modelling

Question 5 (5 marks)

a. Since $V_0 = 8000$, the sum of money Courtney deposited to open her account, was \$8000.

(1 mark)

b. Since $V_{n+1} = 1.035 \times V_n$, the annual interest rate was 3.5% $\left(\text{i.e.}\left(\frac{35}{1000} \times \frac{100}{1}\%\right)\right)$.

(1 mark)

c. $V_0 = 8\,000 \quad V_{n+1} = 1.035 \times V_n$

$$\begin{aligned} V_1 &= 1.035 \times 8\,000 \\ &= 8\,280 \\ V_2 &= 1.035 \times 8\,280 \\ &= 8\,569.80 \\ V_3 &= 1.035 \times 8\,569.80 \\ &= 8\,869.74 \end{aligned}$$

After 3 years, Courtney has \$8 869.74 in her account. **(1 mark)**

d. $V_0 = 8000$

$V_1 = 1.035 \times 8000 = 1.035^1 \times 8000$

$V_2 = 1.035 \times (1.035 \times 8000) = 1.035^2 \times 8000$

$V_3 = 1.035 \times (1.035 \times 1.035 \times 8000) = 1.035^3 \times 8000$

and so on.

So $a = 1.035$ and $b = 8000$. **(1 mark)**

e. Generate the sequence defined by the recurrence relation.
8 000, 8 280, 8 569.8, 8 869.74, 9 180.18, 9 501.49, 9 834.04, 10 178.23.
Note that 8 000 is the opening balance in the account. After 1 year the value of Courtney's savings is \$8 280 and so on. The value first exceeds \$10 000 after 7 years.

(1 mark)

Question 6 (3 marks)

a. $39\,000 \times \$0.32 = \$12\,480$
The car has depreciated over the ten years by \$12 480.
The current value is $\$14\,000 - \$12\,480 = \$1\,520$.

(1 mark)

b. **i.** $C_0 = 14000, \quad C_{n+1} = 0.85 \times C_n$.

(1 mark)

ii. Generate the sequence using technology.
14 000, 11 900, 10 115, 8 597.75, 7 308.09, 6 211.87, 5 280.09, 4 488.08, 3 814.87, 3 242.64, 2 756.24.
Again, remember that 14 000 is the starting value so after 1 year the car's value is \$11 900.
After 10 years the value of the car is \$2 756.24.

(1 mark)

Question 7 (4 marks)

a. Use finance solver.

$N: 6$

$I(\%): 6.2$

$PV: 30\,000$ (positive because the bank gave this to Courtney)

$Pmt: -1200$ (negative because Courtney gave this to the bank)

$FV: ?$

$PpY: 12$

$CpY: 12$

$FV = -23\,648.4524...$ (negative because Courtney still owes the bank this amount)

Courtney still owes $23 648.45.

(1 mark)

b. Courtney has paid the bank $6 \times \$1\,200 = \$7\,200$ in the first six months.
From part **a**. she still owes $23 648.45 so in effect she has paid just
$30 000 – $23 648.45 = $6 351.55 off the principal.
The interest she has paid is $7 200 – $6 351.55 = $848.45.

(1 mark)

c. Start by finding how many payments Courtney makes during the life of the loan.

$N: ?$

$I(\%): 6.2$

$PV: 30\,000$

$Pmt: -1\,200$

$FV: 0$

$PpY: 12$

$CpY: 12$

$N = 26.837739...$

(1 mark)

Over the life of the loan Courtney pays $26.837739... \times \$1\,200 = \$32\,205.29$

So she will have been charged $\$32\,205.29 - \$30\,000 = \$2\,205.29$ in interest over the life of the loan.

(1 mark)

SECTION B - Modules

Module 1 - Matrices

Question 1 (3 marks)

a. Matrix N has order 1×4.

(1 mark)

b. $NM = [25 \times 1800 + 40 \times 1350 + 10 \times 2100 + 55 \times 1950]$
$= [227\,250]$

(1 mark)

c. The information represents the total profit made by the breeder last year in the sale of these four types of puppies.

(1 mark)

Question 2 (2 marks)

a. We need to sum the rows of the dominance matrix.
Abby has dominance over 2 other dogs, Barney has dominance over 1 other, Charlie over 2 others. Drover over 3 others and Ebert over 2 others. So Drover is the most dominant dog.

(1 mark)

b. The two-step dominance matrix is found by squaring the one-step dominance matrix.
The two-step dominance matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix}$$

(1 mark)

Question 3 (7 marks)

a. 30% of the dogs who produce a litter one year are retired the next. **(1 mark)**

b. **i.** $S_1 = TS_0$

$$= \begin{bmatrix} 5 \\ 25 \\ 21 \\ 29 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

Define T and S_0 on your CAS.

(1 mark)

ii. From **i.**,

$$S_1 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.7 & 0.2 & 0 \\ 0.1 & 0.3 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 35 \\ 15 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

The required calculation is $0 \times 10 + 0.7 \times 20 + 0.2 \times 35 + 0 \times 15 = 21$.
(1 mark)

iii. Use trial and error.

From part **i.** we have $S_1 = \begin{bmatrix} 5 \\ 25 \\ 21 \\ 29 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$

Continuing on, we have $S_2 = T^2 S_0$

$$= \begin{bmatrix} 2.5 \\ 14.6 \\ 21.7 \\ 41.2 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

$S_3 = T^3 S_0$

$$= \begin{bmatrix} 1.25 \\ 14.02 \\ 14.56 \\ 50.17 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

$S_4 = T^4 S_0$

$$= \begin{bmatrix} 0.625 \\ 9.236 \\ 12.726 \\ 57.413 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

So the number of dogs producing a litter is 9.236 which is below 10 for the first time. Since S_1 represents the number of dogs at the start of 2016, S_4 represents the number of dogs at the start of 2019.
(1 mark)

c. **i.** The matrix

$$A = \begin{bmatrix} 4 \\ 6 \\ 0 \\ -5 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

tells us that 4 new dogs at the ‘not ready for breeding’ stage are added, 6 new dogs at the ‘producing a litter’ stage are added and 5 dogs who are ‘retired from breeding’ are removed. The net increase is therefore $6+4-5=5$.

(1 mark)

ii. $D_{n+1} = TD_n + A$

$$D_0 = \begin{bmatrix} 10 \\ 20 \\ 35 \\ 15 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix} \qquad \text{(start of 2015)}$$

$$D_1 = TD_0 + A \qquad \text{(start of 2016)}$$
$$= \begin{bmatrix} 9 \\ 31 \\ 21 \\ 24 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

(1 mark)

$$D_2 = TD_1 + A \qquad \text{(start of 2017)}$$
$$= \begin{bmatrix} 8.5 \\ 22.2 \\ 25.9 \\ 33.4 \end{bmatrix} \begin{matrix} N \\ L \\ B \\ R \end{matrix}$$

Note, you must use your matrix $D_1 = \begin{bmatrix} 9 \\ 31 \\ 21 \\ 24 \end{bmatrix}$ from the previous calculation.

So 22 dogs (to the nearest whole number) would be producing a litter at the start of 2017.

(1 mark)

Module 2 - Networks and decision mathematics

Question 1 (4 marks)

a. 70 metres.

(1 mark)

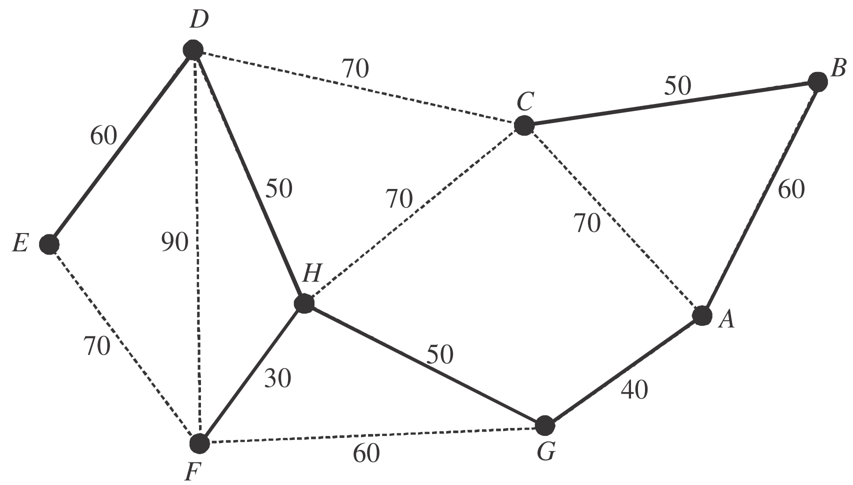
b. A Hamiltonian cycle visits every vertex and starts and finishes at the same vertex. Starting at *A*, the path can be *A B C D E F H G A* or *A B C H D E F G A* or the reverse of each of these. The number of edges visited is 8.

(1 mark)

c. The cleaning machine will be completing an Eulerian trail (i.e. no repeated edges). Six of the vertices of the graph have even degrees i.e. *B*(2), *C*(4), *D*(4), *E*(2), *F*(4) and *H*(4). The other two vertices have odd degrees i.e. *A*(3) and *G*(3).
The cleaning machine will therefore start and finish at either *A* or *G*. The distance between the start and finish is therefore 40 metres.

(1 mark)

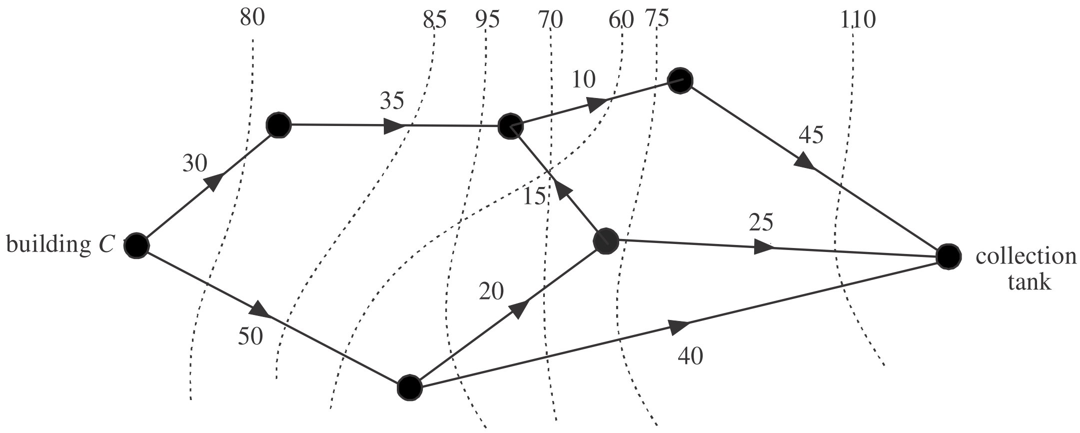
d. A minimum spanning tree is required so we use Prim's algorithm.



(1 mark)

Question 2 (2 marks)

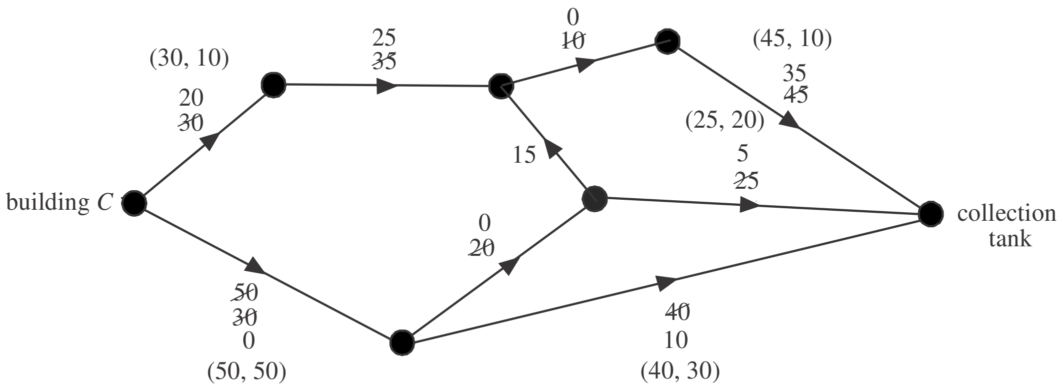
a. Method 1 – inspection



Make some cuts across the system. Be sure that each cut separates building *C* from the collection tank. The minimum cut is 60 and therefore the maximum amount of liquid waste that can flow from building *C* to the collection tank each hour is 60 litres.

(1 mark)

Method 2

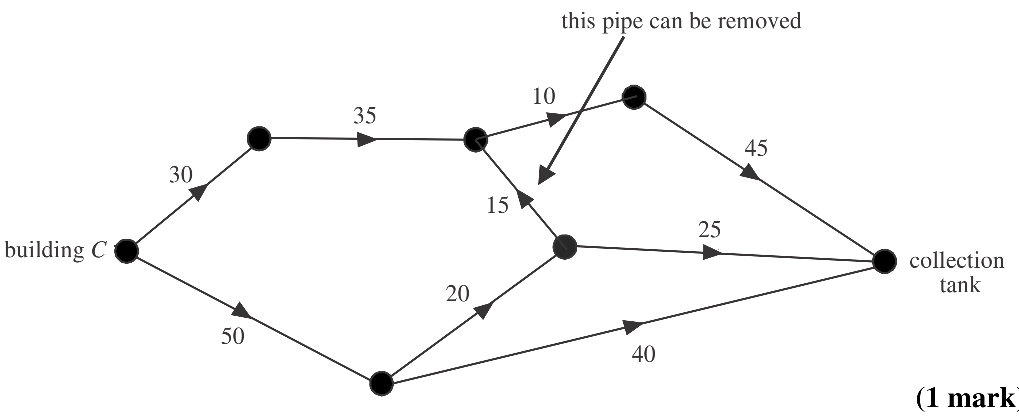


Starting with the upper path, there is a maximum of 10 litres that can flow through per hour. The lower middle path has a maximum of 20 litres that can flow through per hour. The lower bottom path has a maximum of 30 litres that can flow through per hour.

The ordered pairs on the edges give (initial capacity, final flow). The total of the flows out of building *C* is $10+50=60$. The total of the flows into the collection tank is $10+20+30=60$. The maximum flow possible per hour is 60 litres.

(1 mark)

b.



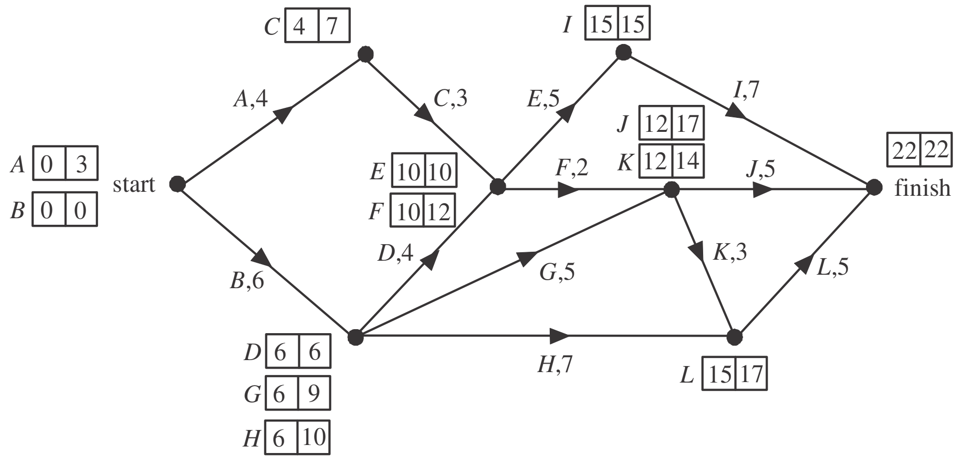
(1 mark)

Question 3 (6 marks)

a. *F* and *G*.

(1 mark)

b. The earliest starting times (EST) are given in the table. Place them on the diagram and then do a backward scan to determine the latest starting times (LST) as shown below.



The LST for activity *C* is 7 weeks.

(1 mark)

c. Using the diagram from part **b**, the minimum completion time is 22 weeks.

(1 mark)

d. Again using the diagram from part **b**. the critical path is *B, D, E, I.*

(1 mark)

e. **i.** Activity *B* lies on the critical path *B D E I*.

B, D, E, I - 22 weeks (critical path)

Looking at other paths and their completion times, we have,

A, C, E, I	-	19 weeks
A, C, F, J	-	14 weeks
A, C, F, K, L	-	17 weeks
B, D, F, J	-	17 weeks
B, D, F, K, L	-	20 weeks
B, G, K, L	-	19 weeks
B, G, J	-	16 weeks
B, H, L	-	18 weeks

If activity *B* is reduced by 2 weeks, the minimum completion time will be 20 weeks.
If activity *B* is reduced by 2 weeks then all the other paths containing activity *B* will also be reduced by 2 weeks. The longest time for these other paths that include activity *B* would be 18 weeks, that is, path *B, D, F, K, L* would take 18 weeks.
If activity *B* was reduced by a further 1 week (a total reduction of 3 weeks) then the minimum completion time would be 19 weeks.
Reducing activity *B* by a further 1 week (a total reduction of 4 weeks) is pointless because the activities on path *A, C, E, I* take 19 weeks.
So the least cost is $3 \times \$500 = \$1\,500$.

(1 mark)

ii. The paths that take 19 weeks are *A, C, E, I* and *B, D, E* and *I*.
So the activities that are critical are *A, B, C, D, E* and *I*. **(1 mark)**

Module 3 - Geometry and measurement

Question 1 (2 marks)

a. The total height of the trophy is $15\text{cm} + 2 \times 7.5\text{cm} = 30\text{cm}$.

(1 mark)

b.

$$\begin{aligned}\text{volume of trophy} &= \text{volume of prism} + \text{volume of sphere}\\ &= 15 \times 15 \times 15 + \frac{4}{3}\pi \times 7.5^3 \quad \text{(formula sheet)}\\ &= 5142.145...\\ &= 5142\text{cm}^3 \quad \text{(to the nearest cubic cm)}\end{aligned}$$

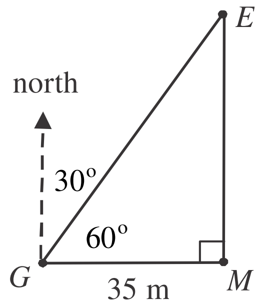
(1 mark)

Question 2 (2 marks)

a. In ΔEGM, $\angle EGM = 60^\circ$

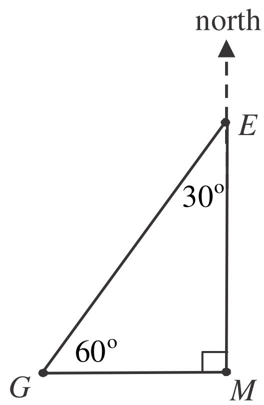
$$\begin{aligned}\cos(60^\circ) &= \frac{35}{EG}\\ \frac{1}{2} \times EG &= 35\\ EG &= 35 \times 2\\ &= 70\end{aligned}$$

The length of the fence is 70 m.



(1 mark)

b. In ΔEGM, $\angle GEM = 30^\circ$.
The bearing of point G from point E
is $180^\circ + 30^\circ = 210^\circ$.



(1 mark)

Question 3 (3 marks)

a. $(CM)^2 = 80^2 + 130^2 - 2 \times 80 \times 130 \times \cos(75\ )$ **(1 mark)**

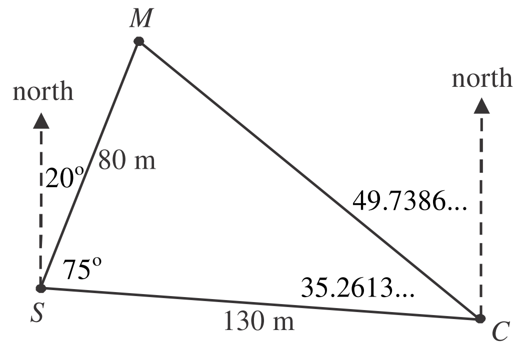
$CM = 133.8527...$

The distance from point C to point M is 134 metres (to the nearest metre).

(1 mark)

b. In ΔCMS, $\dfrac{\sin(\angle MCS)}{80} = \dfrac{\sin(75\ )}{133.8527...}$ (sine rule)

$$\sin(\angle MCS) = \frac{80 \times \sin(75\ )}{133.8527...}$$
$$= 0.5773...$$
$$\angle MCS = \sin^{-1}(0.5773...)$$
$$= 35.2613...$$



Now $85\ - 35.2613...\ = 49.7386...$ and $360\ - 49.7386...\ = 310.2613...\ $.

So the bearing of M from C to the nearest degree is $310\ $.

(1 mark)

Question 4 (3 marks)

a. The latitude of Limburg is 51° North. **(1 mark)**

b. The length of the arc running along the great circle that passes through Royal Ascot and the north pole is given by

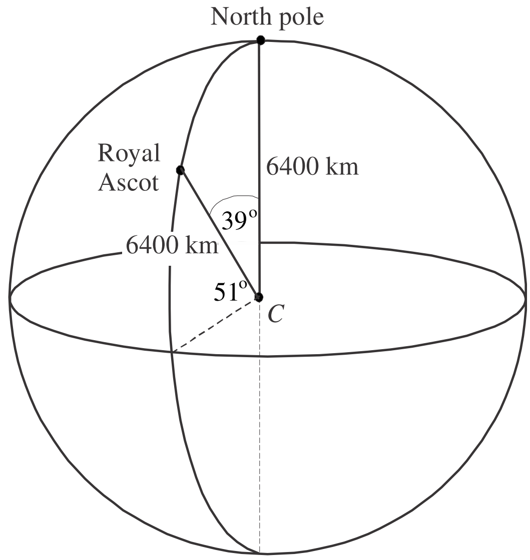
$r \times \dfrac{\pi}{180} \times \theta$ (formula sheet)

$= 6\,400 \times \dfrac{\pi}{180} \times 39$

$= 4\,356.3418...$

$= 4\,356$ km (to the nearest km)

(1 mark)



c. Since Auckland is located at 175°E and Royal Ascot is located at 1°W, Auckland is 13 hours **ahead** of Royal Ascot. So 3:15pm on Saturday at Royal Ascot is 4:15am on Sunday in Auckland.

(1 mark)

Question 5 (2 marks)

surface area = area of rectangles + area of larger semi-circle − area of smaller semi-circle

$$= 2 \times 3 \times 1 + \frac{1}{2} \times \pi \times 2.5^2 - \frac{1}{2} \times \pi \times 1.5^2$$ **(1 mark)**

$$= 12.2831...$$

$$= 12\,\text{m}^2 \text{ (to the nearest square metre)}$$

(1 mark)

Module 4 - Graphs and relations

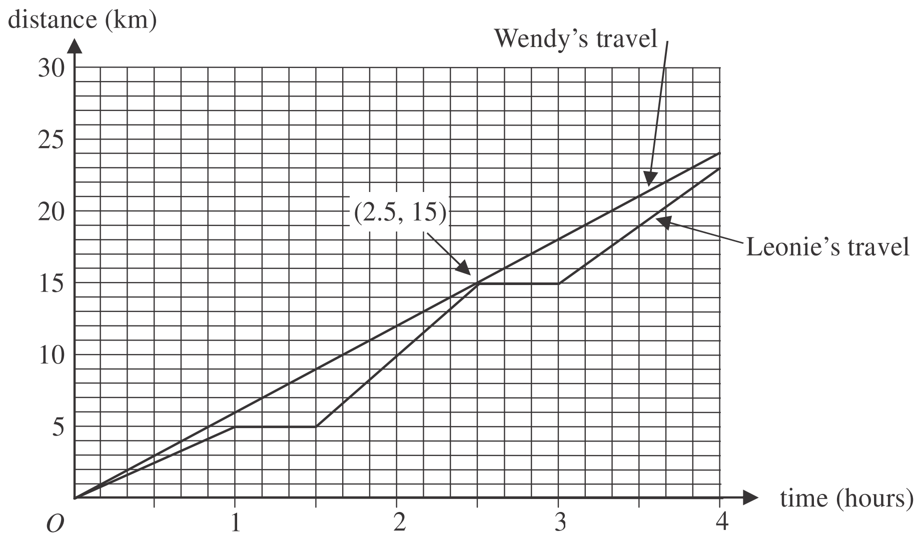
Question 1 (4 marks)

a. Leonie travels $5+10+8=23\text{km}$. **(1 mark)**

b. Leonie was resting for 30 mins $+$ 30 mins i.e 1 hour. **(1 mark)**

c. The fastest speed is indicated by the steepest section of the graph. This occurs between 1.5 and 2.5 hours where the speed was $\frac{\text{rise}}{\text{run}}=\frac{10}{1}$.
The fastest speed was 10 km/hr. **(1 mark)**

d. The easiest way to do this question is to draw the graph showing Wendy's travel. This is given by a straight line with a gradient of 6 (because she travels 6 km every hour) starting at the origin.

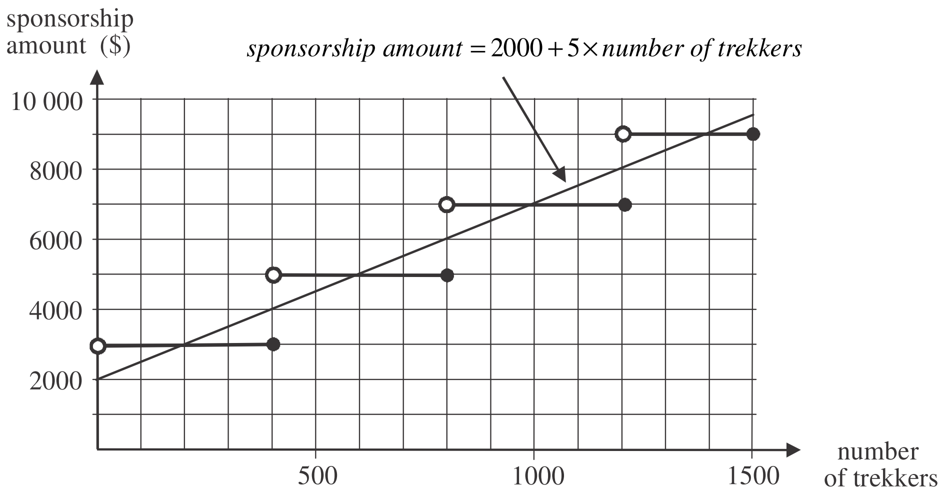


The graphs of Leonie's and Wendy's travel intersect at the point (2.5, 15).
So the two women meet up again 2.5 hours into the session. **(1 mark)**

Question 2 (3 marks)

a. On the step graph shown, the second step from the left coincides with a sponsorship amount of \$5 000. The left hand endpoint of this step however is **<u>not</u>** included. This means that if 400 trekkers participate, the sponsorship amount will be \$3 000 (i.e. the right hand endpoint of the lower step **<u>is</u>** included).
So there needs to be 401 trekkers in order for the charity to receive \$5 000 from the insurance company. **(1 mark)**

b.



(1 mark)

c. For the insurance company, using the step graph, we see that when 1 200 trekkers participate, the sponsorship amount will be \$7 000.
For the bank, using the given rule,

$$\begin{aligned} \textit{sponsorship amount} &= 2\,000 + 5 \times 1\,200 \\ &= \$8\,000 \end{aligned}$$

So the bank contributes \$1 000 more than the insurance company.

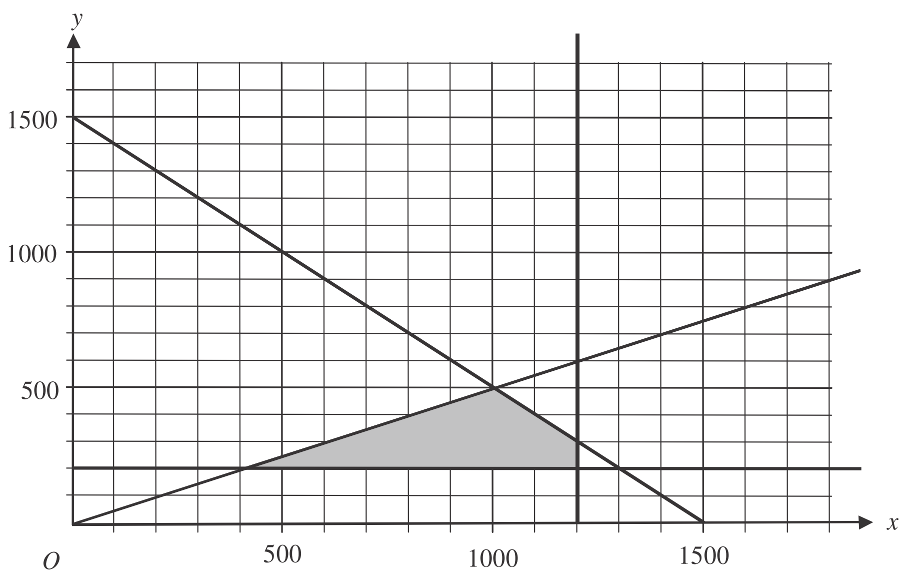
(1 mark)

Question 3 (5 marks)

a. $y \leq \frac{1}{2}x$ or $y \leq \frac{x}{2}$

(1 mark)

b.



(1 mark)

c. From the graph in part **b.**, we see that the maximum number of 60 km trekkers who can participate is 500.

(1 mark)

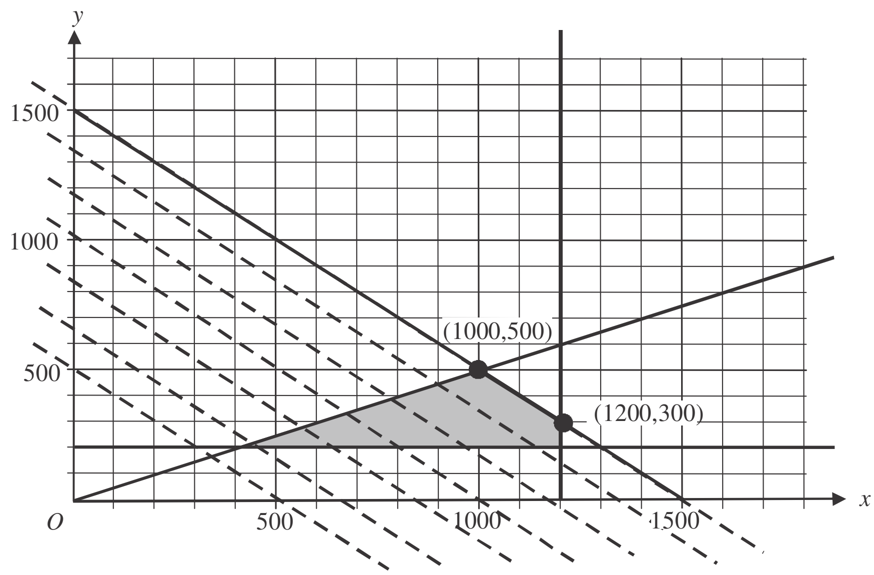
d. **i.** The objective function $A = 500x + 500y$ can be rearranged.

$$500y = -500x + A$$

$$y = -\frac{500x}{500} + \frac{A}{500}$$

$$y = -x + \frac{A}{500}$$ (remember that A is a constant i.e. the total amount raised)

The gradient of the objective function is – 1.
Place your ruler on an angle so that the line it creates has a gradient of -1 as indicated by the dotted lines below.



The sliding dotted lines all have a gradient of -1.

Maintaining this slope, move your ruler to the right.
The last points of contact between the ruler and the shaded (feasible) region occur along the constraint with boundary line given by $x + y = 1\,500$ between the points $(1000, 500)$ and $(1\,200, 300)$ as indicated by the heavy line in the diagram.
Along this line segment, the maximum amount can be found. For example, using the point $(1\,000, 500)$,

$$\begin{aligned} A &= 500x + 500y \\ &= 500 \times 1\,000 + 500 \times 500 \\ &= 750\,000 \end{aligned}$$

Double-check this using the point $(1\,200, 300)$,

$$\begin{aligned} A &= 500 \times 1\,200 + 500 \times 300 \\ &= 750\,000 \end{aligned}$$

The maximum amount that can be raised is \$750 000.

(1 mark)

ii. Again, looking at the heavy line shown in part **d. i**., where the maximum amount raised can be found, we see that the minimum number of 30 km trekkers who participate must be 1 000.

(1 mark)