Neap

Trial Examination 2021

Suggested solutions

QCE Specialist Mathematics Units 3&4

Paper 2 – Technology-active

SECTION 1 – MULTIPLE-CHOICE QUESTIONS

	A	B	C	D
1.	○	○	○	●
2.	●	○	○	○
3.	●	○	○	○
4.	○	●	○	○
5.	○	○	●	○
6.	○	○	○	●
7.	○	○	○	●
8.	○	●	○	○
9.	○	○	○	●
10.	○	○	○	●

ABN 43 634 499 791 91a Orrong Cres, Caulfield North VIC 3161 Tel: (03) 9639 4318

QUESTION 1 D

Let $u = \cos(x)$ so $du = -\sin(x)dx$.

Bounds occur when $x = \frac{\pi}{2},\ u = 0$ and when $x = \frac{\pi}{6},\ u = \frac{\sqrt{3}}{2}$.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2(x)\sin^3(x)dx = -\int_{\frac{\sqrt{3}}{2}}^{0} u^2\left(1-u^2\right)du$$

$$= \int_{0}^{\frac{\sqrt{3}}{2}} u^2\left(1-u^2\right)du$$

QUESTION 2 A

$$\text{confidence interval} = 460 \pm 1.96 \times \frac{45}{\sqrt{25}}$$

$$= (442.36,\ 477.64)$$

QUESTION 3 A

$$\frac{dSA}{dt} = \frac{dSA}{dr} \times \frac{dr}{dt}$$

$$= 8\pi r \times 0.5$$

$$= 4\pi r$$

$$= 32\pi \text{ cm}^2 \text{ s}^{-1}$$

QUESTION 4 B

The net forces are 4 N west and *a* N north.

$$\text{mass} \times \text{acceleration} = 0.6 \times 15$$

$$= 9$$

$$\sqrt{a^2 + 16} = 9$$

$$a^2 + 16 = 81$$

$$a^2 = 65$$

$$a = \sqrt{65} \text{ N}$$

QUESTION 5 C

$$2x - \frac{2ydy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Implicitly differentiating again:

$$\frac{d^2y}{dx^2} = \frac{y - \frac{xdy}{dx}}{y^2}$$

$$= \frac{y - x \times \frac{x}{y}}{y^2}$$

$$= \frac{y^2 - x^2}{y^3}$$

$$= -\frac{7}{y^3}$$

QUESTION 6 D

All coefficients are real, so $1 - 3i$ is also a root.

Roots:

$1 + 3i$, $1 - 3i$, α.

Sum of roots:

$$1 + 3i + 1 - 3i + \alpha = -1$$

$$2 + \alpha = -1$$

$$\alpha = -3$$

Product of roots:

$$(1+3i)(1-3i)\alpha = -a$$

$$10\alpha = -a$$

$$a = -10\alpha$$

$$= 30$$

QUESTION 7 D

Distance travelled in each axis:

$(3 \times 4, 3 \times 2, 3 \times 2) = (12, 6, 6)$

Distance travelled:

$$\sqrt{12^2 + 6^2 + 6^2} = \sqrt{216}$$

$$\approx 14.7 \text{ m}$$

QUESTION 8 B

As AB passes through the centre of the sphere, AB is a diameter.

$$\begin{aligned}\text{AB} &= \sqrt{4^2+1^2+1^2}\\ &= 4.24\text{ cm}\end{aligned}$$

Hence, the radius is 2.12 cm.

QUESTION 9 D

$$\begin{aligned}\int \frac{6}{\sqrt{1-4x^2}}\,dx &= \frac{6}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx\\ &= 3\arcsin\left(\frac{x}{\frac{x}{2}}\right)+C\\ &= 3\arcsin(2x)+C\end{aligned}$$

QUESTION 10 D

$$\begin{aligned}V &= \pi\int_0^a x^{1.4}dx\\ &= \pi\left[\frac{x^{2.4}}{2.4}\right]_0^a\\ &= \pi\left(\frac{a^{2.4}}{2.4}\right)\end{aligned}$$

Where the volume of the solid of revolution is 9 cubic units:

$$\begin{aligned}a^{2.4} &= \frac{9\times 2.4}{\pi}\\ 2.4\ln(a) &= \ln\left(\frac{9\times 2.4}{\pi}\right)\\ a &= e^{\frac{1}{2.4}\ln\left(\frac{9\times 2.4}{\pi}\right)}\\ &= 2.23\end{aligned}$$

SECTION 2

QUESTION 11 (5 marks)

a)

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\cos\theta\sin^3\theta \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\end{aligned}$$

[2 marks]

1 mark for correct expansion of binomial.

1 mark for simplification by recognising $i^2 = -1$ and $i^3 = -i$.

b)

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

[1 mark]

1 mark for correct response.

c) Equating imaginary parts of 11a) and 11b):

$$\begin{aligned}\sin(3\theta) &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

[2 marks]

1 mark for equating imaginary parts.

1 mark for final response.

Note: Consequential on answers to Questions 11a) and 11b).

QUESTION 12 (6 marks)

a) Mean:

$$\begin{aligned}\frac{1}{\lambda} &= 10 \\ \lambda &= \frac{1}{10} \\ &= 0.1\end{aligned}$$

Therefore, $f(x) = 0.1e^{-0.1x}$.

[2 marks]

1 mark for determining λ.

1 mark for $f(x) = 0.1e^{-0.1x}$.

b)

$$\begin{aligned}P(8 < x < 12) &= \int_8^{12} 0.1e^{-0.1x}\,dx \\ &= e^{-0.1(8)} - e^{-0.1(12)} \\ &= 0.1481\end{aligned}$$

[2 marks]

1 mark for correct integral.

1 mark for final response.

Note: Consequential on answer to Question 12a).

c) $\int_0^k 0.1e^{-0.1x}dx = 1 - e^{-0.1k}$

Therefore, $e^{-0.1k} = 0.5$ and $k = 6.93$.

[2 marks]
1 mark for correct integral.
1 mark for final response.

QUESTION 13 (5 marks)

Differentiating implicitly:

$$2x+1=\left(3y^2+6y+2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{2x+1}{3y^2+6y+2}$$

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

When $y = 0$:

$$\begin{aligned} x^2+x &= 0+0+0+2 \\ x^2+x-2 &= 0 \\ (x+2)(x-1) &= 0 \\ x &= 1 \text{ or } -2 \end{aligned}$$

When $x = 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{2(1)+1}{0+0+2}\times 4 \\ &= 6 \qquad \left(\text{since } \frac{dx}{dt}=4\right)\end{aligned}$$

When $x = -2$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{2(-2)+1}{0+0+2}\times 4 \\ &= -6\end{aligned}$$

Therefore, $\frac{dy}{dt} = 6$ when $x = 1$, and $\frac{dy}{dt} = -6$ when $x = -2$.

[5 marks]
1 mark for correct implicit differentiation.

1 mark for finding x values when y = 0.

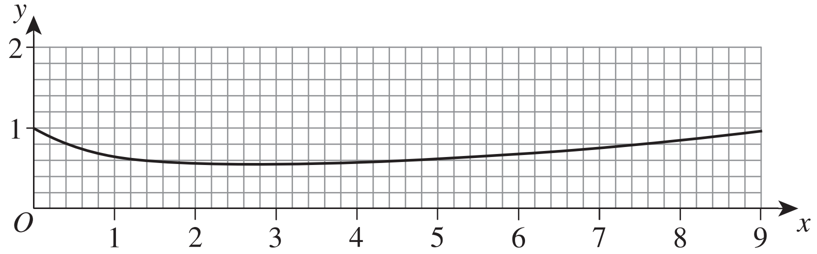
1 mark for chain rule for $\frac{dy}{dt}$.

1 mark for finding $\frac{dy}{dt}$ *for x = 1.*

1 mark for finding $\frac{dy}{dt}$ *for x = –2.*

QUESTION 14 (6 marks)

a)



[2 marks]
1 mark for correct intercept.
1 mark for correct shape.

b)
$$\int_0^8 f(x)dx \approx \frac{2}{3}\left(f(0)+4f(2)+2f(4)+4f(6)+f(8)\right)$$
$$=\frac{2}{3}\left(1+4(0.55)+2(0.54)+4(0.64)+0.82\right)$$
$$=5.1066\ldots$$
$$\approx 5.11$$

[4 marks]
2 marks for using rule correctly.
1 mark for correct y-values.
1 mark for correct calculations.
Note: Consequential on answer to Question 14a).

QUESTION 15 (5 marks)

a) $\mu = 6.42$

$$\frac{\sigma}{\sqrt{96}} = 0.14$$
$$\sigma = 1.37$$

[1 mark]
1 mark for stating μ and calculating σ.

b)
$$\bar{x} = \frac{624}{96}$$
$= 6.5$ hours per battery

Margin of error for 95% confidence interval:

$1.96 \times 0.14 = 0.2744$

$\text{interval} = 6.5 \pm 0.2744$

$= (6.2256,\ 6.7744)$

[3 marks]
1 mark for finding the mean.
1 mark for using the correct formula for a 95% confidence interval.
1 mark for correct interval.

c) The 95% confidence interval calculated in 15b) means that, over a long period of time, 95% of the confidence intervals calculated will contain the true mean for the lifetime of this brand of battery.

[1 mark]
1 mark for providing a suitable outline.

QUESTION 16 (6 marks)

a)

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 140 \\ 0.07 & 0 & 0 \\ 0 & 0.19 & 0 \end{bmatrix}$$

[1 mark]
1 mark for correct 3×3 Leslie matrix.

b)

$$\mathbf{P}_0 = \begin{bmatrix} 1500 \\ 150 \\ 15 \end{bmatrix}$$

$$\mathbf{LP}_0 = \begin{bmatrix} 2100 \\ 105 \\ 28.5 \end{bmatrix}$$

Therefore, there are 2100 females in the first age group after one year.

[2 marks]
1 mark for finding $\boldsymbol{LP}_0$.
1 mark for final response.

c)

$$\mathbf{L}^4\mathbf{P}_0 = \begin{bmatrix} 0 & 0 & 140 \\ 0.07 & 0 & 0 \\ 0 & 0.19 & 0 \end{bmatrix}^4 \begin{bmatrix} 1500 \\ 150 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3910.2 \\ 195.51 \\ 53.067 \end{bmatrix}$$

Therefore, the total number of females is 4158.78.

Since females are 55% of the population:

$$\text{total population} = \frac{4158.78}{55} \times 100$$

$$= 7561.4181\ldots$$

$$\approx 7561$$

[3 marks]
1 mark for finding $\boldsymbol{L}^4\boldsymbol{P}_0$.
1 mark for the total number of females.
1 mark for the total population.

QUESTION 17 (6 marks)

a) $$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$

$$=\pi r^2\times\frac{dh}{dt}$$

$$=9\pi\frac{dh}{dt}$$

$$0.48\pi-0.6h\pi=9\pi\frac{dh}{dt}$$

$$0.48-0.6h=9\frac{dh}{dt}$$

Therefore, $75\frac{dh}{dt}=4-5h$.

[2 marks]

1 mark for correct expression for $0.48\pi-0.6h\pi$ and correctly using the chain rule.

1 mark for final response.

b) When $t=0$, $h=0.2$:

$$4-5h=75\frac{dh}{dt}$$

$$dt=\frac{75}{4-5h}dh$$

$$t=-15\ln(4-5h)+C$$

Substituting in $t=0$, $h=0.2$:

$$0=-15\ln(3)+C$$

$$C=15\ln(3)$$

Thus, $t=-15\ln\left(4-5h\right)+15\ln(3)$.

When $h=0.5$:

$$t=-15\ln(4-2.5)+15\ln(3)$$

$$=10.39\ldots$$

$$\approx 10.4 \text{ minutes}$$

[4 marks]

1 mark for substituting h = 0.2.

1 mark for correct integral.

1 mark for finding C.

1 mark for substituting h = 0.5 and final response.

QUESTION 18 (6 marks)

a) $a = v\dfrac{dv}{dx}$

$$= -\frac{10g + v}{10}$$

$$\frac{1}{10}dx = -\frac{v}{10g + v}dv$$

$$\frac{1}{10}x = 10g\ln(v + 10g) - v + C$$

When $x = 0$, $v = U$:

$$0 = 10g\ln(U + 10g) - U + C$$

$$C = U - 10g\ln(U + 10g)$$

$$\frac{1}{10}x = 10g\ln(v + 10g) - v + U - 10g\ln(U + 10g)$$

When $x = H$ (maximum height), $v = 0$:

$$\frac{1}{10}H = 10g\ln(10g) + U - 10g\ln(U + 10g)$$

$$H = 10U + 100g\ln\left(\frac{10g}{U + 10g}\right)$$

$$= 10U - 100g\ln\left(\frac{U + 10g}{10g}\right)$$

[3 marks]

1 mark for using $a = v\dfrac{dv}{dx}$ and integrating.

1 mark for finding C.

1 mark for substituting in v = 0 and calculating maximum height.

b) $a = \dfrac{dv}{dt}$

$= -\dfrac{10g + v}{10}$

$\displaystyle\int \frac{1}{10g + v} dv = -\frac{1}{10}\int dt$

$\ln(10g + v) = -\dfrac{1}{10}t + C$

When $t = 0$, $v = U$:

$\ln(10g + U) = C$

$\ln(10g + v) = -\dfrac{1}{10}t + \ln(10g + U)$

Let T be the time taken to reach maximum height, so when $t = T$, $v = 0$.

$\ln(10g) = -\dfrac{1}{10}T + \ln(10g + U)$

$\dfrac{1}{10}T = \ln\left(\dfrac{10g + U}{10g}\right)$

$T = 10\ln\left(\dfrac{10g + U}{10g}\right)$

[3 marks]

1 mark for using $a = v\dfrac{dv}{dt}$.

1 mark for finding C.

1 mark for substituting in v = 0 and calculating T.

Note: Consequential on answer to Question 18a).

QUESTION 19 (5 marks)

a) Losing teams

$$\mathbf{D} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \text{Winning teams} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

[1 mark]

1 mark for inserting the correct values into the matrix template.

b)

$$\mathbf{D}+\frac{1}{2}\mathbf{D}^2+\frac{1}{3}\mathbf{D}^3=\begin{bmatrix} 2 & \frac{7}{3} & 3 & \frac{5}{3} & \frac{15}{6} & \frac{15}{6} \\ \frac{3}{2} & \frac{2}{3} & \frac{5}{3} & \frac{2}{3} & \frac{7}{6} & \frac{7}{6} \\ 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{5}{6} & \frac{5}{6} \\ \frac{7}{3} & \frac{5}{3} & \frac{13}{6} & \frac{2}{3} & \frac{3}{2} & \frac{3}{2} \\ \frac{7}{2} & \frac{8}{3} & \frac{23}{6} & \frac{11}{6} & \frac{3}{2} & \frac{7}{3} \\ \frac{7}{2} & \frac{8}{3} & \frac{23}{6} & \frac{11}{6} & \frac{7}{3} & \frac{3}{2} \end{bmatrix}$$

Adding up the rows produces:

$$\begin{bmatrix} \frac{38}{3} \\ \frac{41}{6} \\ \frac{14}{3} \\ \frac{59}{6} \\ \frac{47}{3} \\ \frac{47}{3} \end{bmatrix}$$

Therefore, the final rank of the teams, from first to last, is team 5 and 6 in equal first place, then team 1, team 4, team 2, team 3.

[3 marks]
1 mark for calculating correct dominance matrix.
1 mark for adding up rows correctly.
1 mark for correct final ranking.

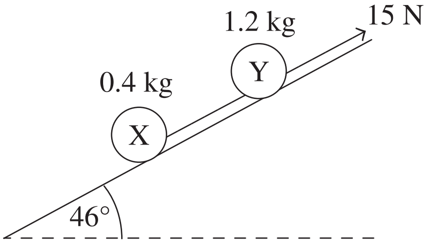
c) *For example:*

The ranking model might not be perfectly accurate over a long period of time, since this is only a snapshot of competition between these teams.

[1 mark]
1 mark for providing a valid limitation.

QUESTION 20 (5 marks)

a)



[1 mark]

1 mark for a drawing diagram that correctly shows the information provided in the question.

b) Resolving forces upwards parallel to the ramp:

$$15-\left(0.4g+1.2g\right)\sin 46°=(0.4+1.2)a$$

$$a=\frac{(15-1.6\times 9.8\times \sin 46°)}{1.6}$$

$$=2.33\text{ m s}^{-2}$$

[2 marks]

1 mark for correct equation by resolving forces.

1 mark for final response.

c) $$a=\frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

$$=2.33$$

$$\frac{1}{2}v^2=2.33x+C$$

When $v=0$ and $x=0$, $C=0$.

$$v=\sqrt{2\times 2.33\times 0.85}$$

$$=1.99\text{ m s}^{-1}$$

[2 marks]

1 mark for $a=\frac{d\left(\frac{1}{2}v^2\right)}{dx}$.

1 mark for final response.