Neap

Trial Examination 2023

Suggested Solutions

QCE Mathematical Methods Units 3&4

Paper 2

SECTION 1 – MULTIPLE CHOICE QUESTIONS

	A	B	C	D
1.	●	○	○	○
2.	○	●	○	○
3.	○	○	○	●
4.	○	○	●	○
5.	○	○	○	●
6.	○	○	●	○
7.	●	○	○	○
8.	○	●	○	○
9.	○	●	○	○
10.	○	○	●	○

QUESTION 1 A

$$\begin{aligned}\log_5\left(x^2\right)-3\log_5(x) &= \log 5\left(x^2\right)-\log_5\left(x^3\right)\\ &= \log_5\left(\frac{x^2}{x^3}\right)\\ &= \log_5\left(\frac{1}{x}\right)\\ &= -\log_5 x\end{aligned}$$

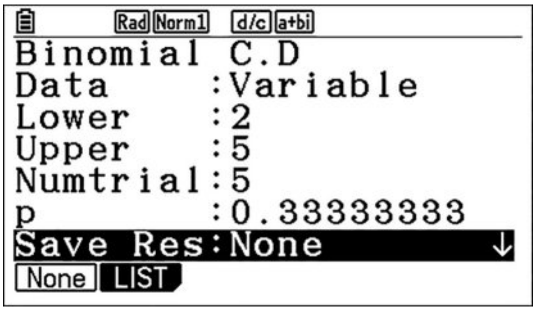
QUESTION 2 B

$$\begin{aligned} y &= 9x-3x^3\\ y' &= 9-9x^2\\ y'' &= -18x\\ \therefore y'' &= -18x = 0\\ x &= 0\end{aligned}$$

Therefore, $x = 0$ is a point of inflection.

QUESTION 3 D

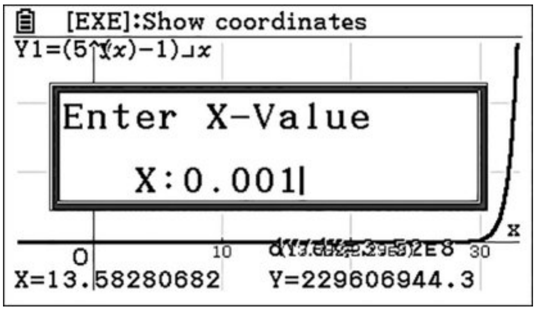
Using a graphics calculator: Statistics, DIST, BINOMIAL, Bcd.

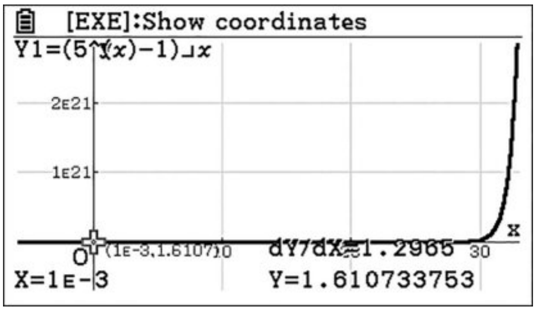


$P(X \geq 2) \approx 0.54$

QUESTION 4 C

Using a graphics calculator: Graph, Trace, $x = 0.001$.





QUESTION 5 D

$$x(t) = \int v(t)\,dt$$
$$= \int 2t^3 - 3t\,dt$$
$$= \frac{1}{2}t^4 - \frac{3}{2}t^2 + c$$

Since $x(0) = 1$:

$$1 = \frac{1}{2}\times 0^4 - \frac{3}{2}\times 0^2 + c$$
$$c = 1$$

Therefore, $x(t) = \frac{t^4}{2} - \frac{3t^2}{2} + 1$.

QUESTION 6 C

Let the known sides be a and b.

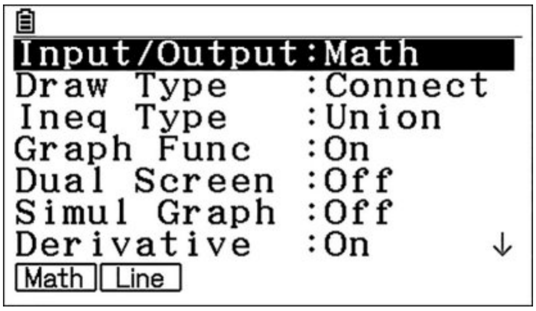
Let A be the angle opposite to side a, and B be the angle opposite to side b.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin A}{5} = \frac{\sin 20}{2}$$
$$\sin A = 0.8551$$
$$A = 58.8°$$

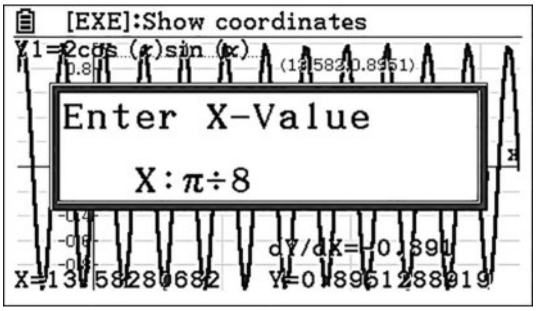
However, the ambiguity of the sine rule applies here as the unknown angle is opposite to side a, which is longer than side b (the side opposite the known angle of 20°). Thus, $A \approx 180 - 58.8 = 121.2°$ is another possible solution.

QUESTION 7 A

Using a graphics calculator: Derivative mode: On.



Using a graphics calculator: Graph, Trace.



$\frac{dy}{dx} \approx 1.41$

QUESTION 8 B

$\hat{p} = 0.51$

The distribution of $\hat{p}$ should be approximately normally distributed with a mean of p and standard deviation, σ, of $\sqrt{\frac{p(1-p)}{n}}$.

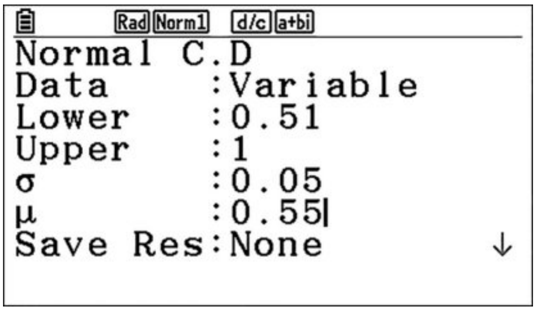
$p = 0.55$

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

$$= \sqrt{\frac{0.55(1-0.55)}{100}}$$

$$\approx 0.05$$

Using a graphics calculator: Statistics, DIST, NORM, Ncd.



$P(\hat{p} > 0.51) \approx 0.79$

Option **B** is the closest to this value.

QUESTION 9 B

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

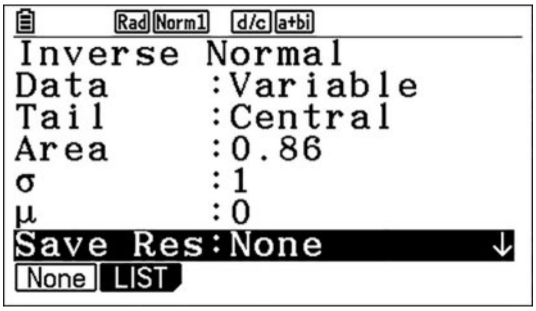
$$4 = \int_a^b f(x)dx - 5$$

$$\int_a^b f(x)dx = 9$$

QUESTION 10 C

Firstly, the z-score needs to be determined.

Using a graphics calculator: Statistics, DIST, NORM, InvN.



$z \approx 1.48$

$p = 0.175$

$$\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= \sqrt{\frac{0.175(1-0.175)}{80}}$$

$$= 0.0425$$

The confidence interval is:

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = (0.175 - 0.0628, 0.175 + 0.0620)$$

$$= (0.11, 0.24)$$

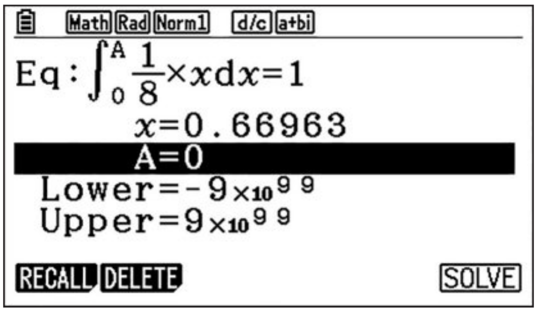
SECTION 2

QUESTION 11 (4 marks)

a) $\int_0^a \frac{1}{8} x\, dx = 1$

$\int_0^a x\, dx = 8$

Using a graphics calculator: Equation, Solver.



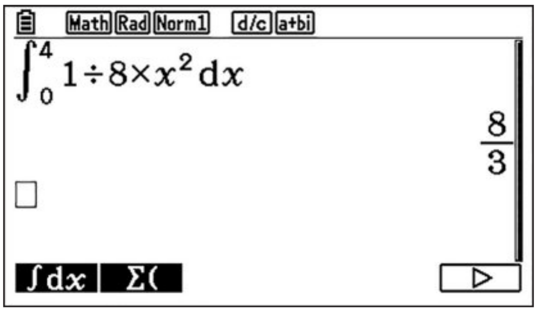
$a = 4$

[1 mark]
1 mark for determining the value of a.

b) $E(X) = \int_{-\infty}^{\infty} x(fx)dx$

$= \int_0^4 \frac{1}{8} x^2 dx$

Using a graphics calculator: Run-Matrix, MATH, $\int dx$.



$E(X) = \frac{8}{3}$

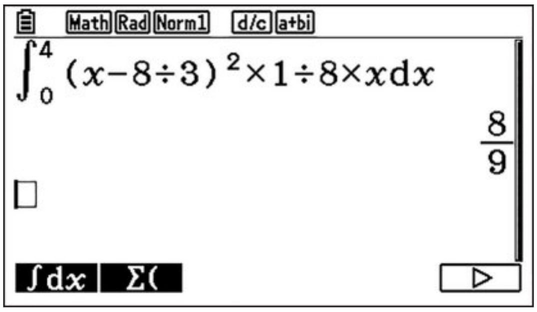
[1 mark]
1 mark for determining the expected value.
*Note: Consequential on answer to **Question 11a)**.*

c) $\sigma = \sqrt{Var(X)}$

$$Var(X) = \int_{-\infty}^{\infty} \left(x - E(X)^2 \times f(x)\right) dx$$

$$= \int_0^4 \left(\left(x - \frac{8}{3}\right)^2 \times \frac{1}{8}x\right) dx$$

Using a graphics calculator: Run-Matrix, MATH, $\int dx$.



$$Var(X) = \frac{8}{9}$$

$$\sigma = \sqrt{Var(X)}$$
$$= \sqrt{\frac{8}{9}}$$
$$= 0.94$$

[2 marks]

1 mark for using an appropriate method. Note: This includes taking the square root of the variance.

1 mark for determining the standard deviation. Note: Accept values in the range 0.94 ± 0.01.

QUESTION 12 (5 marks)

a) $\text{pH} = -\log_{10} \text{H}^+$

$$= -\log_{10} 0.003$$
$$= 2.52$$

The solution is acidic.

[2 marks]

1 mark for determining the pH. Note: Accept values in the range 2.52 ± 0.05.

1 mark for identifying that the solution is acidic.

b) $\text{pH} = 2.52 + 5$

$$= 7.52$$

$$7.52 = -\log_{10} \text{H}^+$$

$$\text{H}^+ = 10^{-7.52}$$
$$\approx 3.02 \times 10^{-8} \text{ mol L}^{-1}$$

[1 mark]

1 mark for determining the concentration of hydrogen ions.

*Note: Consequential on answer to **Question 12a)**.*

c)
$$\begin{aligned} \text{pH} &= -\log_{10} \text{H}^+ \\ \text{pH} - 3.5 &= -\log_{10} k\,\text{H}^+ \\ \text{pH} - 3.5 &= -\log_{10} k - \log_{10} \text{H}^+ \\ -3.5 &= -\log_{10} k \\ k &= 10^{3.5} \\ &\approx 3162.3 \end{aligned}$$

[2 marks]
1 mark for setting up an appropriate equation to solve for k.
1 mark for determining the value of k.

QUESTION 13 (6 marks)

a) $a(t) = v'(t) = -2e^{-2t} + 1$

[1 mark]
1 mark for determining the acceleration function.

b)
$$\begin{aligned} d(t) &= \int v(t)\,dt \\ &= \int e^{-2t} + t\,dt \\ &= -\frac{e^{-2t}}{2} + \frac{t^2}{2} + c \end{aligned}$$
$d(0) = 0$
$$\begin{aligned} \therefore -\frac{e^{-2\times 0}}{2} + \frac{0^2}{2} + c &= 0 \\ -\frac{1}{2} + c &= 0 \\ c &= \frac{1}{2} \end{aligned}$$
Therefore:
$$d(t) = -\frac{e^{-2t}}{2} + \frac{t^2}{2} + \frac{1}{2}$$

[2 marks]
1 mark for integrating v(t).
1 mark for determining the displacement function.

c) The equation $v(t) = e^{-2t} + t = 0$ has no solution because $t > 0$ implies that both terms are positive. Therefore, there is no time at which the velocity is 0 μm s^{-1}. Thus, there are no times when the particle is at rest.

[2 marks]
1 mark for determining that the particle is never at rest.
1 mark for providing adequate justification.

d) The minimum is reached when $a(t) = -2e^{-2t} + 1 = 0$.

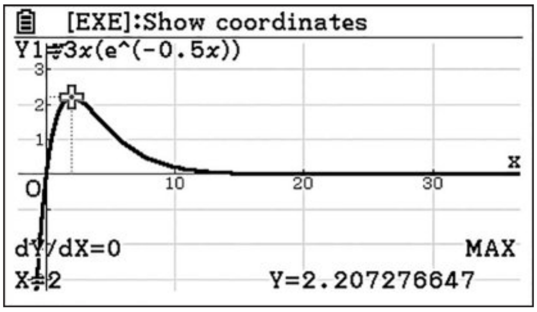
Thus:

$$\begin{aligned} -2e^{-2t} + 1 &= 0 \\ -2e^{-2t} &= -1 \\ e^{-2t} &= \frac{1}{2} \\ t &= -\frac{1}{2}\ln\frac{1}{2} \\ &= 0.35 \text{ s} \end{aligned}$$

[1 mark]
1 mark for determining the value of t at the minimum.

QUESTION 14 (7 marks)

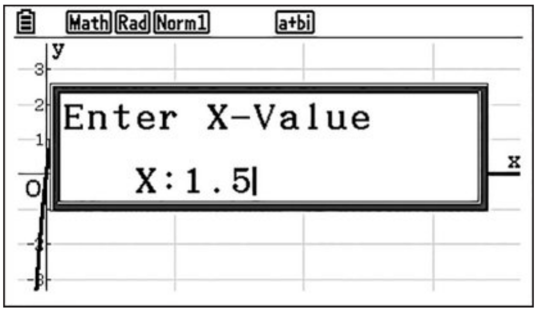
a) Using a graphics calculator: Graph, G-Solv, MAX.



The maximum instantaneous rate of change is 2.2 and occurs when $t = 2$ hours.

[1 mark]
1 mark for determining the maximum instantaneous rate of change and the time when it occurs.

b) Using a graphics calculator: Graph, G-Solv, X-CAL.



$t_1 \approx 0.7$

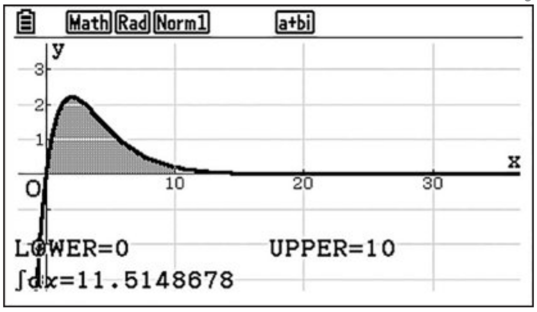
$t_2 \approx 4.3$

$$\begin{aligned} \text{duration} &= t_2 - t_1 \\ &= 4.3 - 0.7 \\ &= 3.6 \text{ hours} \end{aligned}$$

[2 marks]
1 mark for determining at least one endpoint.
1 mark for estimating the duration of time.

c) total change $= \int_0^{10} 3te^{-0.5t}\,dt$

Using a graphics calculator: Graph, G-Solv, $\int dx$.



total mass = 11.5 nanograms

[2 marks]
1 mark for setting up an appropriate definite integral.
1 mark for determining the total mass.

d) average rate of change $= \dfrac{\text{total change}}{\text{duration}}$

$= \dfrac{1}{10}\int_0^{10} 3te^{-0.5t}\,dt$

$= 1.15$ nanograms per hour

[1 mark]
1 mark for determining the average rate of change.

e) *For example, any one of:*

- The model may overestimate the total body exposure; this is because the function never reaches 0 due to the asymptote. *(To estimate the total body exposure, an upper limit for the integral would need to be chosen arbitrarily and used to calculate the area under the curve.)*
- The model is only accurate for estimation purposes. It simplifies and smooths out the rate of change. It has only one maximum, while in reality the graph should have more peaks and troughs.

[1 mark]
1 mark for making an appropriate and plausible comment regarding the accuracy of the model based on graphical evidence.
Note: Accept other appropriate responses.

QUESTION 15 (3 marks)

Determining the value of γ gives:

$$C(26.5) = -e^{\frac{26.5-10}{5}} + \gamma = 0$$
$$\gamma = e^{\frac{26.5-10}{5}}$$
$$= 27.113$$

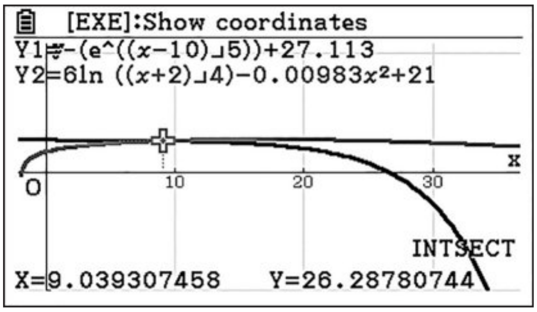
Determining the value of α gives:

$$B'(t) = \frac{6}{t+2} - 2\alpha t$$

Since $t = 26.5 - 10 = 16.5$ maximises the population of octopuses:

$$B'(16.5) = \frac{6}{16.5+2} - 2\alpha \times 16.5 = 0$$
$$\alpha = 0.001$$

Using a graphics calculator to graph the functions of $C(t)$ and $B(t)$: Graph, G-Solv, INTSECT.



The functions intersect at 9.1 weeks. Therefore, the two populations will be the same size at 9.1 weeks.

[3 marks]

1 mark for determining the value of γ.

1 mark for determining the value of α. Note: Accept values between 0.001 and 0.0097.

1 mark for determining when the populations will be the same size. Note: Accept values in the range 9.1 ± 0.1.

QUESTION 16 (3 marks)

$n = 150$

$\hat{p} = \dfrac{46}{150}$

Let E be the margin of error:

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= z\sqrt{\frac{\frac{46}{150}\left(1-\frac{46}{150}\right)}{150}}$$

$$= z\sqrt{\frac{46\times 104}{150^3}}$$

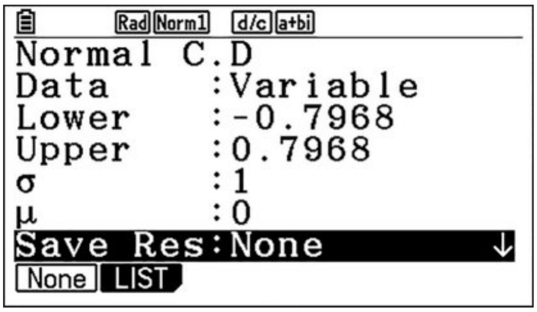
$$= 0.0377z$$

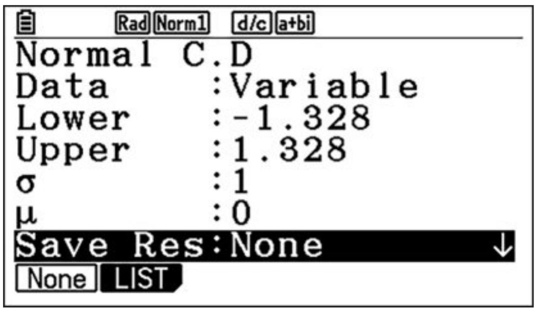
$0.03 < E < 0.05$

Thus, $0.03 < 0.0377z < 0.05$.

$\therefore 0.7968 < z < 1.3280$

Using a graphics calculator to determine the probabilities: DIST, NORM, Ncd.





$P(-0.7968 < z < 0.7968) \approx 0.57$

$P(-1.3280 < z < 1.3280) \approx 0.82$

Therefore, the confidence levels that can be achieved are between 57% and 82%

[3 marks]

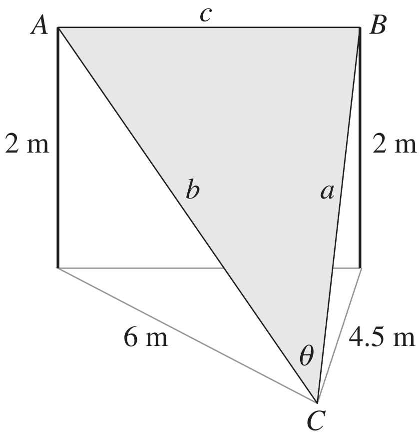
1 mark for using an appropriate method to solve for the z-values (for example, substituting the margin of error formula).

1 mark for determining the two z-values. Note: Accept values in the ranges 0.7968 ± 0.01 and 1.3280 ± 0.01.

1 mark for determining the range of confidence level. Note: Accept values in the ranges $57 \pm 1\%$ and $82 \pm 1\%$.

QUESTION 17 (5 marks)

Let A, B and C be the vertices of the shade cloth, and the angle at C of the triangle on the ground be θ.



Determining the size of the angle θ gives:

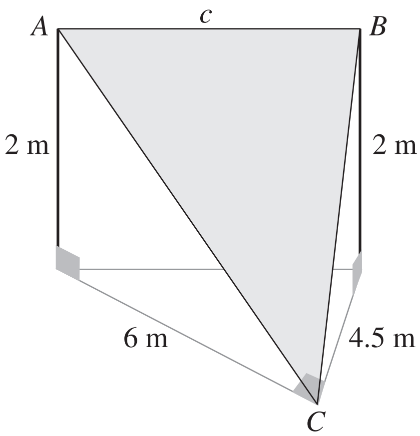
$$\text{area} = \frac{1}{2}bc\sin A$$

$$13.5 = \frac{1}{2} \times 6 \times 4.5\sin\theta$$

$$\sin\theta = 1$$

$$\theta = 90°$$

This means that the base triangle is a right-angled triangle.



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Determining the lengths of a, b and c gives:

$a^2 = 4.5^2 + 2^2$

$a = \sqrt{24.25}$

$b^2 = 2^2 + 6^2$

$b = \sqrt{40}$

$c^2 = 4.5^2 + 6^2$

$c = \sqrt{56.25}$

Determining the angle at C gives:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$\cos(C) = \frac{c^2 - a^2 - b^2}{-2ab}$$

$$\cos(C) = \frac{56.25 - 24.25 - 40}{-2 \times \sqrt{24.25} \times \sqrt{40}}$$

$$C = 82.62°$$

[5 marks]

1 mark for drawing at least one accurate diagram.

1 mark for determining that the base triangle is a right-angled triangle.

1 mark for determining the side lengths of the shade cloth.

1 mark for determining the value of C.

1 mark for showing clear and logical organisation of working.

Note: Accept follow-through errors.

QUESTION 18 (4 marks)

Let E represent the random variable for English.

Let H represent the random variable for History.

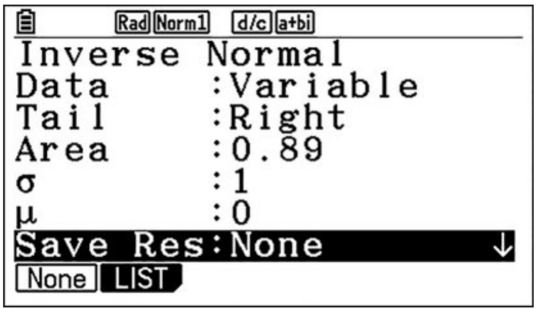
$P(E > 45) = 0.89$

$P(E < 70) = 0.8$

$P(H > 65) = 0.5$

$P(45 < H < 85) = 0.82$

Using a graphics calculator to determine the z-scores: Statistics, DIST, NORM, InvN.



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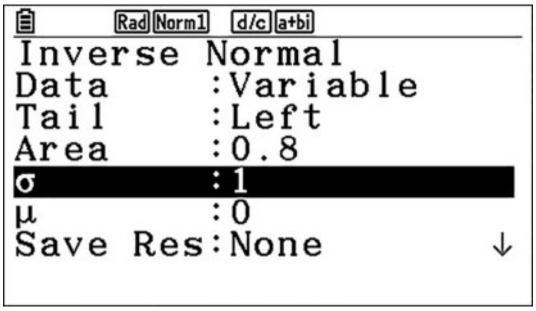


$P(E > 45) = 0.89$, right tail, $z = -1.23$

$P(E < 70) = 0.8$, left tail, $z = 0.841$

$P(H > 65) = 0.5$, right tail, $z = 0$

Determining the mean and standard deviation for English gives the following.

$$z = \frac{E - \mu_E}{\sigma_E}$$

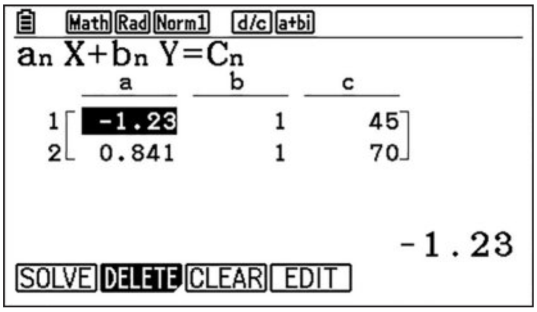
Using the two z-scores for E:

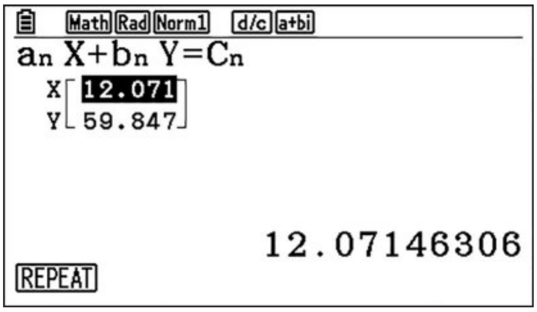
$$-1.23 = \frac{45 - \mu_E}{\sigma_E}$$

$$0.841 = \frac{70 - \mu_E}{\sigma_E}$$

Therefore, $-1.23\sigma_E + \mu_E = 45$ and $0.841\sigma_E + \mu_E = 70$.

Using a graphics calculator: Equation, Simultaneous.





$\sigma_E = 12.1$

$\mu_E = 59.8$

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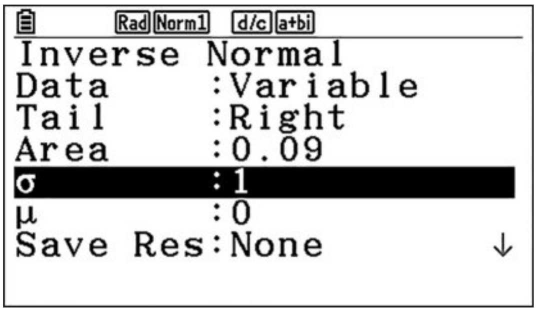
Determining the mean and standard deviation for History gives the following.

The mean for History is 65.

45 and 85 are both 20 away from the mean.

Since the normal distribution is symmetrical about the mean, $P(65 < H < 85) = 0.41$. Also, $P(H > 85) = 0.09$.

Using a graphics calculator: Statistics, DIST, NORM, InvN.



right tail, $z = 1.34$

Therefore:

$$z = \frac{H - \mu_H}{\sigma_H}$$

$$1.34 = \frac{85 - 65}{\sigma_H}$$

$$\sigma_H = 14.93$$

Comparing Vanessa's two scores using z-scores:

English:

$$z = \frac{E - \mu_E}{\sigma_E} = \frac{80 - 59.8}{12.1} = 1.67$$

History:

$z = 1.34$

Therefore, Vanessa scored comparatively better in English.

[4 marks]

1 mark for determining the mean and standard deviation for English.

1 mark for determining the mean and standard deviation for History.

1 mark for determining the z-scores for Vanessa's English and History scores.

1 mark for determining that Vanessa scored comparatively better in English.

QUESTION 19 (3 marks)

It is assumed that the probability that the production line will be shut down is ≤ 0.30. This will only occur if at least one of the tested tins is contaminated.

Let X represent the number of tins that are contaminated.

$X \sim Bi(2, p)$, where p is unknown.

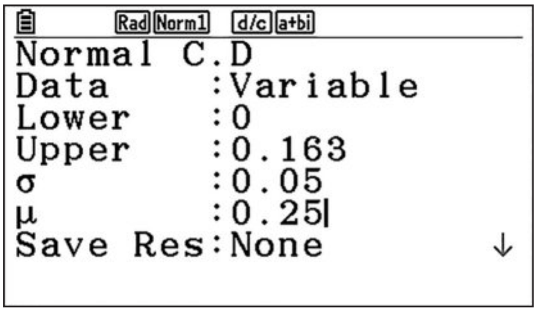
$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \binom{2}{0} p^0 (1-p)^2 \\ &= 1 - (1-p)^2 \end{aligned}$$

Supposing that $1-(1-p)^2 < 0.3$:

$$\begin{aligned} -(1-p)^2 &< -0.7 \\ (1-p)^2 &> 0.7 \\ 1-p &> \sqrt{0.7} \\ -p &> \sqrt{0.7} - 1 \\ p &< 1 - \sqrt{0.7} \\ &< 0.1633 \end{aligned}$$

Let P represent the random variable for the probability that a tin is contaminated for a particular production batch. P is normally distributed with a mean of 0.25 and a standard deviation of 0.05.

Using a graphics calculator: Statistics, DIST, NORM, Ncd.



$P(0 \leq P \leq 0.1633) \approx 0.04$

[3 marks]

1 mark for determining an appropriate expression in terms of p for $P(X \geq 1)$.

1 mark for determining the maximum possible value of p.

1 mark for determining the probability. Note: Accept values in the range 0.038–0.042.

QUESTION 20 (5 marks)

Firstly, the graphs show the rate of the number of subscribers per week.

Let $p(t)$ represent the number of subscribers for the politics channel.

Let $f(t)$ represent the number of subscribers for the fashion channel.

Identifying the equations for $p'(t)$ and $f'(t)$ gives the following.

Politics channel:

$$\begin{aligned} p'(t) &= a\sin\left(b(t-c)\right)+d \\ &= 500\sin\left(\frac{1}{2}t\right)+100 \end{aligned}$$

Justification from the graph:

- The equilibrium is 100; therefore, $d = 100$.
- The amplitude is 200; therefore, $a = 500$.
- No phase shift has occurred since $p'(t)$ is modelled using a sine function; therefore, $c = 0$.
- There is a difference of π between the equilibrium and the maximum. Therefore, the period is 4π and $b = \frac{1}{2}$.

The two channels had the same subscriber rate on days 9 and 31.

- 9 days is approximately 1.29 weeks.
- 31 days is approximately 4.43 weeks.

$p'(1.29) \approx 400$ subscribers per week

$p'(4.43) \approx 500$ subscribers per week

Fashion channel:

$f'(t)$ intersects the points (1.29, 400) and (4.43, 500).

Thus:

$$f'(1.29) = 200\log_a(1.29+b)+300 = 400$$

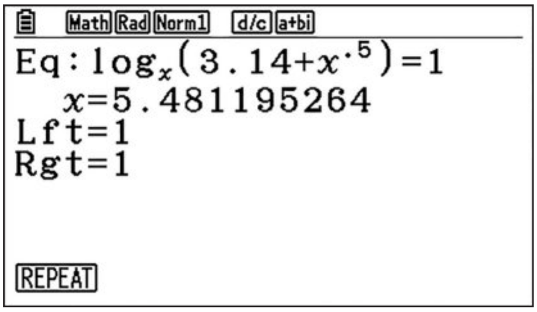
$$f'(4.43) = 200\log_a(4.43+b)+300 = 500$$

$$\begin{aligned} 200\log_a(1.29+b)+300 &= 400 \\ \log_a(1.29+b) &= 0.5 \\ 1.29+b &= a^{0.5} \\ b &= a^{0.5}-1.29 \end{aligned}$$

Substituting this into the other equation gives:

$$\begin{aligned} 200\log_a\left(4.43+a^{0.5}-1.29\right)+300 &= 500 \\ \log_a\left(3.14+a^{0.5}\right) &= 1 \end{aligned}$$

Using a graphics calculator: Equation.



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(continued)

$a \approx 5.48$

Also, $b = 5.48^{0.5} - 1.29 \approx 1.05$.

Thus, $f'(t) = 200\log_{5.48}(t + 1.05) + 300$.

Letting $s(t)$ represent the total number of subscribers gives:

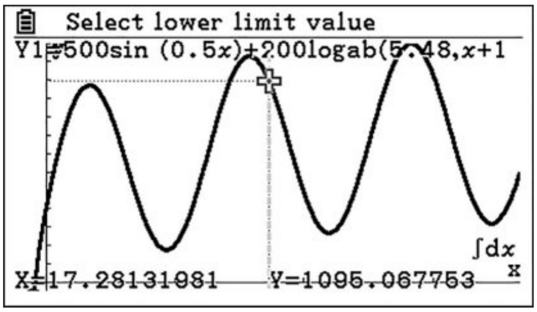
$$s'(t) = p'(t) + f'(t)$$

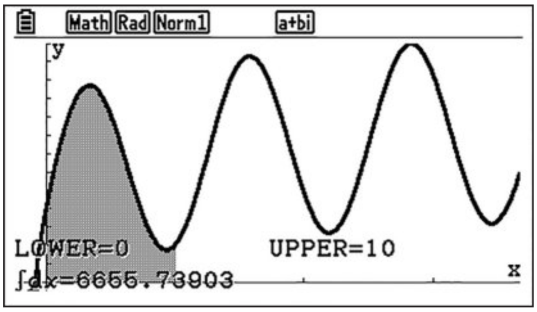
$$= 500\sin\left(\frac{1}{2}t\right) + 100 + 200\log_{5.48}(t + 1.05) + 300$$

$$= 500\sin\left(\frac{1}{2}t\right) + 200\log_{5.48}(t + 1.05) + 400$$

The net change in the number of subscribers $= \int_0^{10} s'(t)dt$.

Using a graphics calculator: Graph, G-Solv, $\int dx$.





net change $= 6655.7$ subscribers

While Felix's estimate of more than 7000 subscribers is close to this figure, his claim is not reasonable.

[5 marks]

1 mark for determining the trigonometric function and the coordinates of the points of intersection.

Note: Accept y-coordinates in the ranges of 400 ± 1 and 500 ± 1.

1 mark for determining the logarithmic function unknowns.

1 mark for recognising the need to integrate the rate of change functions to determine the total change.

1 mark for determining the total change. Note: Accept values in the range of 6655 ± 10.

1 mark for making an accurate comparison based on appropriate mathematical evidence.

Note: Accept follow-through errors.