Neap

Trial Examination 2023

HSC Year 12 Mathematics Extension 2

Solutions and Marking Guidelines

SECTION I

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 1 **A** $\begin{pmatrix}3\\5\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=5$ $\cos\theta=\dfrac{5}{\sqrt{3^2+5^2+(-2)^2}\times\sqrt{1^2}}$ $\theta=35.8°$	MEX–V1 Further Work with Vectors MEX12–3 Band E2
Question 2 **A** Rewriting the equation gives: $\arg\left(\dfrac{z-1}{z+2i}\right)=\arg(z-1)-\arg(z+2i)$ $=\pi$ This represents the line segment between points (1, 0) and (0, –2). These are the only possible positions of point z such that the difference between the argument of $z-1$ and the argument of $z+2i$ is π.	MEX–N2 Using Complex Numbers MEX12–4 Band E3
Question 3 **D** The contrapositive of the statement is: 'If my teacher did not give me a detention, then I did complete my homework.' Hence, the converse of the contrapositive is: 'If I do complete my homework, then my teacher will not give me a detention.'	MEX–P1 The Nature of Proof MEX12–2, 12–8 Band E3
Question 4 **D** $(a+bi)(2-i)=3+i$ $2a+b=3\Rightarrow 4a+2b=6$ $-a+2b=1$ $\therefore 5a=5$ $\therefore a=1,\ b=1$	MEX–N1 Introduction to Complex Numbers MEX12–1, 12–4 Band E2
Question 5 **A** As $\lvert z\rvert=1,\ z+1$ is the long diagonal of a rhombus with side length of one unit. Hence, $\arg(z+1)=\dfrac{\theta}{2}$, and $\cos\dfrac{\theta}{2}=\dfrac{\frac{1}{2}\lvert z+1\rvert}{1}$. $\therefore\lvert z+1\rvert=2\cos\left(\dfrac{\theta}{2}\right)$	MEX–N2 Using Complex Numbers MEX12–1, 12–5 Bands E3–E4

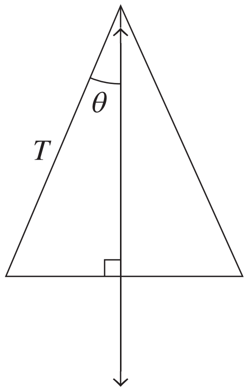
Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 6 **C** **C** is correct. The line passes through point (5, 2, 1); hence, the fixed point of the vector equation is $\begin{pmatrix}5\\2\\1\end{pmatrix}$. The line is parallel to the *x*–*y* plane and *x*–*z* plane; thus, the direction vector is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ as it represents movement in the *x*-direction only. **A** is incorrect. This equation represents movement in the *x*-direction and *y*-direction. This would make the line parallel to the *x*–*y* plane but not the *x*–*z* plane. **B** is incorrect. This equation represents movement in the *y*-direction and *z*-direction. This would make the line parallel to the *y*–*z* plane but not the *x*–*y* plane or the *x*–*z* plane. **D** is incorrect. This equation represents movement in the *x*-direction and *z*-direction. This would make the line parallel to the *x*–*z* plane but not the *x*–*y* plane.	MEX–V1 Further Work with Vectors MEX12–3 Band E3
Question 7 **A** Given that $v^2 = 20 - 16x - 4x^2$: $\frac{1}{2}v^2 = 10 - 8x - 2x^2$ $\therefore a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x - 8$ $= -4(x+2)$ Hence, the particle moves in a simple harmonic motion about the centre $x = -2$, $n = 2$. Therefore, its period is $\frac{2\pi}{2} = \pi$. Given that $v^2 = 20 - 16x - 4x^2$: $0 = -4(x^2 + 4x - 5)$ $= -4(x+5)(x-1)$ Hence, the turning points are at *x* = –5 and *x* = 1 and the amplitude is 3.	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Band E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 8 **C** $ma = mg - kv$ $a = g - kv$ $\frac{dv}{dt} = 10 - \frac{v}{2}$ $= \frac{20 - v}{2}$ $\frac{dt}{dv} = \frac{2}{20 - v}$ $\int_0^t dt = 2\int_0^v \frac{1}{20 - v} dv$ $t = -2\left[\ln\lvert 20 - v\rvert\right]_0^v$ $= 2\left[\ln 20 - \ln\lvert 20 - v\rvert\right]$ $\frac{t}{2} = \ln\left\lvert\frac{20}{20 - v}\right\rvert$ $\frac{20}{20 - v} = e^{\frac{t}{2}}$ $20 - v = 20e^{-\frac{t}{2}}$ $v = 20\left(1 - e^{-\frac{t}{2}}\right)$	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Band E3
Question 9 **A** **A** is correct. This statement is true for all values of $a, b, c, d \in \mathbb{R}$. **B** is incorrect. This statement has the following counter-example. $a = 10,\ b = 9,\ a > b$ $c = 5,\ d = -10,\ c > d$ $a - c = 5,\ b - d = 19,\ a - c < b - d$ **C** is incorrect. This statement has the following counter-example. $a = 10,\ b = 4,\ a > b$ $c = -1,\ d = -2,\ c > d$ $ac = -10,\ bd = -8,\ ac < bd$ **D** is incorrect. This statement has the following counter-example. $a = 10,\ b = 9,\ a > b$ $c = 5,\ d = 3,\ c > d$ $\frac{a}{c} = 2,\ \frac{b}{d} = 3,\ \frac{a}{c} < \frac{b}{d}$	MEX–P1 The Nature of Proof MEX12–2, 12–8 Band E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 10 **C** Since $x = \sqrt{t-2}$, $t = x^2 + 2$. Thus: $y = \frac{1}{2-t}$ $= -\frac{1}{x^2}$ $\because t > 2$, $x > 0$, $y < 0$ Hence, the solution should only include the fourth quadrant of the graph.	MEX–V1 Further Work with Vectors MEX12–1, 12–3 Band E3

SECTION II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) Assume that $n \in \mathbb{Z}^+$ and $\sqrt{3n+1}$ is rational. $3n+1 = \frac{p^2}{q^2}$, $p, q \in \mathbb{Z}$ $n = \frac{p^2}{3q^2} - \frac{1}{3}$ (contradiction) If $n \in \mathbb{Z}^+$, then $\sqrt{3n+1}$ is always irrational, as n will not always be an integer.	MEX–P1 The Nature of Proof MEX12–1, 12–2, 12–8 Bands E2–3 • Provides the correct solution . . . . . . 2 • Uses the correct method 1
(b) $\alpha + \beta = \sqrt{3}cis\left(\frac{\pi}{3}\right) + \sqrt{3}cis\left(-\frac{\pi}{3}\right)$ $= 2\cos\left(\frac{\pi}{3}\right) \times \sqrt{3}$ $= \sqrt{3}$ $\alpha\beta = \sqrt{3}cis\left(\frac{\pi}{3}\right) \times \sqrt{3}cis\left(-\frac{\pi}{3}\right)$ $= 3$ $\therefore x^2 - \sqrt{3}x + 3 = 0$	MEX–N2 Using Complex Numbers MEX12–1, 12–4 Bands E2–3 • Provides the correct solution . . . . . . 2 • Makes some progress using the sum and product of the roots. OR • Equivalent merit 1
(c) Let $u = \sin^{-1} 3x$ and $\frac{du}{dx} = \frac{1}{\sqrt{\frac{1}{9} - x^2}}$. $\frac{dv}{dx} = 1$, $v = x$ $\int \sin^{-1} 3x dx = x \sin^{-1} 3x - \int \frac{x}{\sqrt{\frac{1}{9} - x^2}} dx$ $x = x \sin^{-1} 3x + \sqrt{\frac{1}{9} - x^2} + C$	MEX–C1 Further Integration MEX12–5 Band E3 • Provides the correct solution . . . . . . 2 • Makes some progress applying integration by parts 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d)  $2(T\cos\theta) = 500$ $250 = T\cos\theta$ $250 = 326\cos\theta$ $\cos\theta = \frac{250}{326}$ $\theta = 39.9°$ $\therefore$ maximum angle $\approx 80°$	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E3 • Provides the correct solution . . . . . .2 • Uses the correct equation for the vertical force AND tension in one rope. 1
(e) (i) Assuming that the two lines intersect gives: $2-\lambda = -1+\mu$ $3 = \lambda + \mu$ (x-coordinate) $1+\lambda = 1-2\mu$ $0 = \lambda + 2\mu$ (y-coordinate) $\mu = -3, \lambda = 6$ However, substituting these values into the z-coordinate gives: $1-\lambda = -5$ $\mu = -3$ $1-\lambda \neq \mu$	MEX–V1 Further Work with Vectors MEX12–3 Band E3 • Solves all THREE simultaneous equations to show the inconsistency 1
(ii) $\overrightarrow{PQ} = \begin{pmatrix} -1+\mu \\ 1-2\mu \\ \mu \end{pmatrix} - \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 1-\lambda \end{pmatrix}$ $= \begin{pmatrix} -3+\mu+\lambda \\ -2\mu-\lambda \\ -1+\mu+\lambda \end{pmatrix}$	MEX–V1 Further Work with Vectors MEX 12–3 Band E3 • Provides the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) $\begin{pmatrix}-3+\mu+\lambda\\-2\mu-\lambda\\-1+\mu+\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix}=0$ $\therefore 4\mu+3\lambda=4$ $\begin{pmatrix}-3+\mu+\lambda\\-2\mu-\lambda\\-1+\mu+\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=0$ $\therefore 6\mu+4\lambda=4$ $\therefore \lambda=4,\ \mu=-2$ $\therefore P:(-2, 5, -3), Q:(-3, 5, -2)$ $\lvert\overrightarrow{PQ}\rvert=\sqrt{2}$ *Note: Consequential on answer to* ***Question 11(e)(ii)***.	MEX–V1 Further Work with Vectors MEX12–3 Bands E3–E4 • Provides the correct solution . . . . . . 3 • Uses the dot product of $\overrightarrow{PQ}$ and the direction vectors of L_1 and L_2 to find points P and Q . . . . 2 • Makes some progress towards finding points P and Q 1
(f) Given that $2\lvert z-1\rvert=\lvert z-4\rvert$: $4\lvert z-1\rvert^2=\lvert z-4\rvert^2$ $4(z-1)\overline{(z-1)}=(z-4)\overline{(z-4)}$ $4(z-1)(\bar{z}-1)=(z-4)(\bar{z}-4)$ $4z\bar{z}-4z-4\bar{z}+4=z\bar{z}-4z-4\bar{z}+16$ $3z\bar{z}=12$ $\lvert z\rvert^2=4$ $\lvert z\rvert=2$	MEX–N1 Introduction to Complex Numbers MEX12–1, 12–4 Band E3 • Provides the correct solution . . . . . . 2 • Makes some progress using conjugate theorems for complex numbers. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) (i) $\ddot{x} = -\frac{96\,000}{x^2}$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{96\,000}{x^2}$ $\frac{1}{2}v^2 = \int -\frac{96\,000}{x^2}dx$ $v^2 = -192\,000\int x^{-2}dx$ $= \frac{192\,000}{x} + c$ $\because x = 6400,\ v = 8,$ then $c = 34$ At $v = 6.5$: $v = \sqrt{\frac{192\,000}{x} + 34}$ $6.5 = \sqrt{\frac{192\,000}{x} + 34}$ $x = 23\,273$ km $\therefore 23\,273 - 6400 = 16\,873$ km	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Band E3 • Provides the correct solution . . . . . .3 • Provides the correct value of x. . . . .2 • Provides the correct expression of v^2 1
(ii) $\because v = \sqrt{\frac{192\,000}{x} + 34}$ $\lim\limits_{x\to\infty} = \sqrt{\frac{192\,000}{x} + 34}$ $= \sqrt{34}$ km s^{-1} *Note: Consequential on answer to* ***Question 12(a)(i)***.	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Band E3 • Provides the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) $P(1)$: $2(1) > 1 + \frac{1}{3^1} > \frac{1}{3(1)}$ $2 > \frac{4}{3} > \frac{1}{3}$ Therefore, $P(1)$ is true. If $P(k)$ is true, $2k > 1 + \frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^k} > \frac{1}{3k}$. $P(k+1)$: Consider the series: $\frac{1}{3^k+1} + \frac{1}{3^k+2} + \ldots + \frac{1}{3^k+3^k+3^k}$ This series has 2×3^k terms. Since $\frac{1}{3^k} > \frac{1}{3^k+1} > \frac{1}{3^k+3^k+3^k}$ and $\frac{1}{3^k} > \frac{1}{3^k+2} > \frac{1}{3^k+3^k+3^k}$ and so on. $\therefore \frac{1}{3^k}\times(2\times3^k) > \frac{1}{3^k+1} + \frac{1}{3^k+2} + \ldots$ $+ \frac{1}{3^k+3^k+3^k} > \frac{1}{3^k+3^k+3^k}\times(2\times3^k)$ $2 > \frac{1}{3^k+1} + \frac{1}{3^k+2} + \ldots + \frac{1}{3\times3^k} > \frac{2}{3}$ Adding this result to $P(k)$: $2k+2 > 1 + \frac{1}{3} + \ldots + \frac{1}{3^k+3^k+3^k} > \frac{1}{3k} + \frac{2}{3}$ $2(k+1) > 1 + \frac{1}{3} + \ldots + \frac{1}{3(3^k)} > \frac{2k+1}{3k}$ Since $k > 0$, $\frac{2k+1}{3k} > \frac{2k+1}{3k+3} > \frac{1}{3(k+1)}$ $\therefore 2(k+1) > 1 + \frac{1}{3} + \ldots + \frac{1}{3^{k+1}} > \frac{2k+1}{3k} > \frac{1}{3(k+1)}$ As $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$, $P(n)$ is true for $\forall n \in \mathbb{Z}^+$.	MEX–P2 Further Proof by Mathematical Induction MEX12–1, 12–2, 12–7, 12–8 Band E4 • Provides the correct solution . . . . . . 4 • Considers adding 2×3^k extra terms to the series. AND • Provides the correct manipulation of the lower limit of the inequality . . . . 3 • Considers adding 2×3^k extra terms to the series . . . . 2 • Provides the correct proof for $P(1)$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $\because \lvert z\rvert = \lvert\omega\rvert \therefore \left\lvert \frac{z}{\omega} \right\rvert = 1$ $\arg\left(\frac{z}{\omega}\right) = \arg(z) - \arg(\omega)$ $= \frac{\pi}{3}$	MEX–N1 Introduction to Complex Numbers MEX12–1, 12–4 Band E3 • Provides the correct modulus AND argument of $\frac{z}{\omega}$ . . . . .2 • Provides the correct modulus OR argument of $\frac{z}{\omega}$ 1
(ii) $\because \lvert z\rvert = \lvert\omega\rvert$ $\left\lvert \frac{z^3}{\omega^3} \right\rvert = \frac{\lvert z\rvert^3}{\lvert\omega\rvert^3}$ $= 1$ $\arg\left(\frac{z^3}{\omega^3}\right) = \arg(z^3) - \arg(\omega^3)$ $= 3\arg(z) - 3\arg(\omega)$ $= 3\arg\left(\frac{z}{\omega}\right)$ $= \pi$ $\therefore z^3 + \omega^3 = 0$	MEX–N1 Introduction to Complex Numbers MEX12–1, 12–4 Band E3 • Provides the correct solution . . . . . .2 • Makes progress applying de Moivre's theorem. OR • Equivalent merit 1
(d) $\int \frac{x^3+4x^2-2x-33}{x^2-9}dx = \int x + 4 + \frac{7x+3}{x^2-9}dx$ $= \frac{1}{2}x^2 + 4x + \int \frac{A}{x-3} + \frac{B}{x+3}dx$ $7x+3 = A(x+3) + B(x-3)$ $A = 9,\ B = -2$ $\int \frac{x^3+4x^2-2x-33}{x^2-9}dx = \frac{1}{2}x^2 + 4x + \int \frac{9}{x-3} - \frac{2}{x+3}dx$ $= \frac{1}{2}x^2 + 4x + 9\ln\lvert x-3\rvert - 2\ln\lvert x+3\rvert + C$	MEX–C1 Further Integration MEX12–5 Band E3 • Provides the correct solution . . . . . .3 • Provides the correct partial fraction. . . . . .2 • Provides the correct polynomial division 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	
(a) $\because a > b > 0,\ a+b+1 > b+b+1 > 0$ $\therefore a+b+1 > 2b+1$ $(a+b+1)^2 > (2b+1)^2$ $(a+b+1)^2 > 4b^2+4b+1$ $\because 4b^2+4b+1 > 4b^2+4b = 4b(b+1) > 3b(b+1)$ $\therefore (a+b+1)^2 > 3b(b+1)$	MEX–P1 The Nature of Proof MEX12–2, 12–8 Band E3 • Provides the correct proof for $4b^2+4b+1 > 3b(b+1)$. AND • Provides the correct connection between the inequalities . . . . 3 • Provides the correct proof for $(a+b+1)^2 > 4b^2+4b+1$ . . . . . 2 • Provides the correct proof for $a+b+1 > 2b+1$ 1
(b) As $t = \tan\frac{x}{2}$, then $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$. $\frac{dx}{dt} = 2\cos^2\frac{x}{2}$ $= \frac{2}{t^2+1}$ $\int \frac{\cos x\,dx}{4+3\cos x} = \int \frac{\frac{1}{3}(4+3\cos x) - \frac{4}{3}}{4+3\cos x}dx$ $= \frac{1}{3}\int 1 - \frac{4}{4+3\cos x}dx$ $= \frac{x}{3} - \frac{1}{3}\int \frac{4}{4+3\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{t^2+1}dt$ $= \frac{x}{3} - \frac{8}{3}\int \frac{1}{t^2+7}dt$ $= \frac{x}{3} - \frac{8}{3}\left(\frac{1}{\sqrt{7}}\arctan\frac{t}{\sqrt{7}}\right) + C_1$ $= \frac{x}{3} - \frac{8\sqrt{7}}{21}\arctan\frac{\sqrt{7}\tan\frac{x}{2}}{7} + C_2$	MEX–C1 Further Integration MEX12–5 Band E3 • Provides the correct solution . . . . . . 3 • Performs the correct simplification AND factorisation arriving at $\int \frac{1}{t^2+7}dt$ . . . . 2 • Provides the correct algebraic manipulation arriving at $\frac{1}{3}\int 1 - \frac{4}{4+3\cos x}dx$. OR • Equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) $9z^4 - 18z^3 + 5z^2 - 18z + 9 = 0$ $9z^2 - 18z + 5 - 18z^{-1} + 9z^{-2} = 0$ $9\left(z^2 + z^{-2}\right) - 18\left(z + z^{-1}\right) + 5 = 0$ $9(2\cos 2\theta) - 18(2\cos\theta) + 5 = 0$ $18\left(2\cos^2\theta - 1\right) - 36\cos\theta + 5 = 0$ $36\cos^2\theta - 18 - 36\cos\theta + 5 = 0$ $36\cos^2\theta - 36\cos\theta - 13 = 0$ $\cos^2\theta - \cos\theta - \frac{13}{36} = 0$ $\left(\cos\theta - \frac{1}{2}\right)^2 - \frac{1}{9} = 0$ $\left(\cos\theta - \frac{1}{2}\right)^2 = \frac{1}{9}$ As $\cos\theta = \frac{5}{6}$, $\sin\theta = \frac{\pm\sqrt{11}}{6}$, then $z = \frac{5}{6} \pm \frac{\sqrt{11}}{6}i$. As $\cos\theta = \frac{1}{6}$, $\sin\theta = \frac{\pm\sqrt{35}}{6}$, then $z = \frac{1}{6} \pm \frac{\sqrt{35}}{6}i$.	MEX–N2 Using Complex Numbers MEX12–1, 12–4 Bands E3–4 • Provides the correct solutions for all FOUR values of z ....4 • Solves for $\cos\theta$ ....3 • Derives the correct quadratic equation $36\cos^2\theta - 36\cos\theta - 13 = 0$. OR • Equivalent merit ....2 • Makes some progress dividing the polynomial by z^2 and collecting conjugate pairs.................1
(d) (i) $x = 3\sin\theta + 2,\ x - 2 = 3\sin\theta$ $y = 3\cos^2\theta + 1,\ y - 1 = 3\cos^2\theta$ $z = 3\sin\theta\cos\theta + 5,\ z - 5 = 3\sin\theta\cos\theta$ $(x-2)^2 + (y-1)^2 + (z-5)^2 = (3\sin\theta)^2 + \left(3\cos^2\theta\right)^2 + (3\sin\theta\cos\theta)^2$ $= 9\sin^2\theta + 9\cos^2\theta\cos^2\theta + 9\sin^2\theta\cos^2\theta$ $= 9\left(\sin^2\theta + \cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)\right)$ $= 9\left(\sin^2\theta + \cos^2\theta\right)$ $= 9$	MEX–V1 Further Work with Vectors MEX12–1, 12–3 Band E3 • Performs factorisation and uses the Pythagorean identity to simplify the expression $(x-2)^2 + (y-1)^2 + (z-5)^2 = 9$ ..2 • Substitutes to obtain $(3\sin\theta)^2 + \left(3\cos^2\theta\right)^2 + (3\sin\theta\cos\theta)^2$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\because c : (2, 1, 5)$ $\therefore \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $2 = -2 + 2\lambda,\ \lambda = 2$ $1 = -3 + 2\lambda,\ \lambda = 2$ $5 = 3 + \lambda,\ \lambda = 2$ As the value of λ is consistent for all three equations, line L passes through the centre point (2, 1, 5).	MEX–V1 Further Work with Vectors MEX12–1, 12–3 Band E3 • Provides the correct solution 1
(iii) $(-2 + 2\lambda - 2)^2 + (-3 + 2\lambda - 1)^2 + (3 + \lambda - 5)^2 = 9$ $(2\lambda - 4)^2 + (2\lambda - 4)^2 + (\lambda - 2)^2 = 9$ $4\lambda^2 - 16\lambda + 16 + 4\lambda^2 - 16\lambda + 16 + \lambda^2 - 4\lambda + 4 = 9$ $9\lambda^2 - 36\lambda + 36 = 9$ $\lambda^2 - 4\lambda + 4 = 1$ $\lambda^2 - 4\lambda + 3 = 0$ $(\lambda - 1)(\lambda - 3) = 0$ $\lambda = 1,\ \lambda = 3$ $\begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ $\begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$ Therefore, line L and the surface of the sphere intersect at points $(0, -1, 4)$ and $(4, 3, 6)$.	MEX–V1 Further Work with Vectors MEX12–1, 12–3 Bands E3–E4 • Provides the correct solution . . . . . . 2 • Finds the quadratic equation 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 14	
(a) $\int \frac{x+3}{\sqrt{9-8x-x^2}}dx = \int \frac{x+4-1}{\sqrt{9-8x-x^2}}dx$ $= \int \frac{x+4}{\sqrt{9-8x-x^2}}dx - \int \frac{1}{\sqrt{9-8x-x^2}}dx$ $= \sqrt{9-8x-x^2} - \int \frac{1}{\sqrt{25-(x+4)^2}}dx$ $= \sqrt{9-8x-x^2} - \arcsin\left(\frac{x+4}{5}\right) + C$	MEX–C1 Further Integration MEX12–5 Band E5 • Provides the correct solution . . . . . .3 • Derives $\sqrt{9-8x-x^2}$ . . . . .2 • Separates the expression into two fractions. 1
(b) Let $z = rcis\theta$. $\because \operatorname{Im}\left(z+\frac{1}{z}\right)=0$ $\operatorname{Im}\left(rcis\theta+\frac{1}{rcis\theta}\right)=0$ $\therefore r\sin\theta + r^{-1}\sin\theta^{-1} = 0$ $r\sin\theta + r^{-1}\sin(-\theta) = 0$ $r\sin\theta - r^{-1}\sin\theta = 0$ $\sin\theta\left(r-r^{-1}\right)=0$ $\sin\theta = 0,\ r-\frac{1}{r}=0$ $\therefore \operatorname{Im}(z)=0$ or $\frac{r^2-1}{r}=0$ As $\operatorname{Im}(z)\neq 0$, then $r^2=1$, $r=\pm 1$. $\therefore \lvert z\rvert = 1$	MEX–N2 Using Complex Numbers MEX12–1, 12–4 Band E3 • Derives $r=\pm 1$ . . . . .3 • Uses sine as an odd function and derives $\sin\theta\left(r-r^{-1}\right)=0$ . . . . .2 • Applies de Moivre's theorem to derive $r\sin\theta + r^{-1}\sin\theta^{-1} = 0$. OR • Equivalent merit 1
(c) (i) Let $\frac{a+b}{2} \geq \sqrt{ab}$. $\therefore \frac{(x-2)+2}{2} \geq \sqrt{2(x-2)}$ $x \geq 2\sqrt{2(x-2)}$	MEX–P1 The Nature of Proof MEX12–2, 12–8 Band E3 • Provides the correct proof 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let $x = a^2 + 2$. $a^2 + 2 \ge 2\sqrt{2(a^2 + 2 - 2)}$ $a^2 + 2 \ge 2\sqrt{2}a$ $\because a > 0,\ a^2 + 2 > 0$ and $2\sqrt{2}a > 0$ $(a^2 + 2)^2 \ge 8a^2$ $a^4 + 4a^2 + 4 \ge 8a^2$	MEX–P1 The Nature of Proof MEX12–2, 12–7, 12–8 Bands E3–E4 • Provides the correct condition AND proof for $(a^2 + 2)^2 \ge 8a^2$ . . . . .2 • Provides the correct proof for $a^2 + 2 \ge 2\sqrt{2}a$ 1
(d) (i) Let t be the time in hours after the helicopter leaves its base. Path of the helicopter: $\begin{pmatrix} -25 \\ 124 \\ 28 \end{pmatrix} + t\begin{pmatrix} 18 \\ 12 \\ 4 \end{pmatrix}$ Path of the missile: $\begin{pmatrix} -8 \\ -238 \\ 3 \end{pmatrix} + (t-1)\begin{pmatrix} 20 \\ 280 \\ 25 \end{pmatrix}$ Assuming the missile will hit the helicopter: $\begin{pmatrix} -25 \\ 124 \\ 28 \end{pmatrix} + t\begin{pmatrix} 18 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ -238 \\ 3 \end{pmatrix} + (t-1)\begin{pmatrix} 20 \\ 280 \\ 25 \end{pmatrix}$ $-25 + 18t = -8 + 20t - 20,\ t = 1.5$ $124 + 12t = -238 + 280t - 280,\ t = 2.34$ Since the value of t is inconsistent for the x- and y-coordinates, the missile will not collide with the helicopter.	MEX–V1 Further Work with Vectors MEX12–3, 12–7, 12–8 Bands E3–E4 • Shows the inconsistency for the value of t for the x- and y-coordinates . . . . .2 • Finds the vector equations for the paths of the helicopter AND missile 1

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Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Path of the helicopter: $\begin{pmatrix} -25 \\ 124 \\ 28 \end{pmatrix} + t\begin{pmatrix} 18 \\ 12 \\ 4 \end{pmatrix}$ Path of the missile: $\begin{pmatrix} -8 \\ -238 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 20 \\ 280 \\ 25 \end{pmatrix}$ Intersection of missile with the helicopter: $\begin{pmatrix} -25 \\ 124 \\ 28 \end{pmatrix} + t\begin{pmatrix} 18 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ -238 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 20 \\ 280 \\ 25 \end{pmatrix}$ Equating x: $-25 + 18t = -8 + 20\lambda$ $-17 = 20\lambda - 18t$ $-238 = 280\lambda - 252t$ Equating y: $124 + 12t = -238 + 280\lambda$ $362 = 280\lambda - 12t$ Solving simultaneously: $600 = 240t$ $t = 2.5$ $\therefore \lambda = 1.4$ Equating z and checking for consistency: $28 + 4t = 3 + 25\lambda$ $\text{LHS}: 28 + 4t = 38$ $\text{RHS}: 3 + 25\lambda = 38$ Hence, it is possible that the missile may intersect with the helicopter. The total flight time for the helicopter is 2.5 hours. This means the helicopter will intersect with the missile at 10:30 am. The total flight time for the missile is 1.4 hours. This means it will intersect with the helicopter 1 hour and 24 minutes before 10:30 am. Therefore, the missile would need to be fired at 9:06 am to collide with the helicopter.	MEX–V1 Further work with Vectors MEX12–3, 12–7, 12–8 Band E4 • Provides the correct solution . . . . . . 4 • Shows consistency for the values of t and λ for the x, y and z components . . . . 3 • Finds the values of t and λ . . . . 2 • Equates the vector equations 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 15	
(a) (i) Let $\lvert z\rvert = \lvert z+2\rvert$. $a^2+b^2=(a+2)^2+b^2$ $a^2=a^2+4a+4$ $a=-1$ $\therefore x=-1$	MEX–N2 Using Complex Numbers MEX12–1, 12–4 Bands E2–E3 • Provides the correct solution 1
(ii) Let $\lvert z\rvert = 2$. $a^2+b^2=4$ $1+b^2=4$ $b=\pm\sqrt{3}$ $\therefore z_1=2e^{\frac{2\pi}{3}i},\ z_2=2e^{-\frac{2\pi}{3}i}$	MEX–N2 Using Complex Numbers MEX 12–1, 12–4 Band E3 • Provides the correct solution ...... 2 • Finds the value of b 1
(iii) $\arg\left(\dfrac{\upsilon\omega^k}{ki}\right)=\arg\left(2e^{\frac{2\pi}{3}i}\right)+k\arg\left(2e^{-\frac{2\pi}{3}i}\right)-\arg(i)$ $=\dfrac{2\pi}{3}-\dfrac{2k\pi}{3}-\dfrac{\pi}{2}$ $=\dfrac{1-4k}{6}\pi$ $\because \text{Re}\left(\dfrac{\upsilon\omega^k}{ki}\right)=0$ $\dfrac{1-4k}{6}\pi=\pm\dfrac{\pi}{2}$ $k=-\dfrac{1}{2},\ k=1$	MEX–N2 Using Complex Numbers MEX12–1, 12–4 Band E3–E4 • Provides the TWO correct values of k .... 2 • Finds $\arg\left(\dfrac{\upsilon\omega^k}{ki}\right)$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) Let $u = x^n$ and $\frac{du}{dx} = nx^{n-1}$. Let $\frac{dv}{dx} = \cos x$ and $v = \sin x$. $\therefore \int x^n \cos x dx = x^n \sin x - n\int x^{n-1} \sin x dx$ Let $u = x^{n-1}$ and $\frac{du}{dx} = (n-1)x^{n-2}$. Let $\frac{dv}{dx} = \sin x$ and $v = -\cos x$. $\int x^{n-1} \sin x dx = -x^{n-1} \cos x + \int (n-1)x^{n-2} \cos x dx$ $\int x^n \cos x dx = x^n \sin x + n\left(x^{n-1} \cos x - (n-1)I_{n-2}\right)$ $= x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}$	MEX–C1 Further Integration MEX12–5 Band E4 • Provides the correct recurrence relation . . . . 3 • Performs the second integration by parts . . . . 2 • Performs the first integration by parts 1
(ii) $I_0 = \int_0^{\pi} \cos x dx$ $= [\sin x]_0^{\pi}$ $= 0$ $I_2 = \left[x^2 \sin x + 2x \cos x\right]_0^{\pi} - 2(1)I_0$ $= -2\pi$ $I_4 = \left[x^4 \sin x + 4x^3 \cos x\right]_0^{\pi} - 4(3)I_2$ $= -4\pi^3 + 24\pi$	MEX–C1 Further Integration MEX12–5 Band E3 • Provides the correct solution . . . . . . 2 • Calculates $I_0 = 0$. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $P(2): T_1 = 3,\ T_2 = 4(2)^2 - 1 = 15$ $T_2 = \dfrac{3 \times \left(2(2)^2 + 2\right)}{2(2) - 3}$ $= 15$ $\therefore P_2$ is true. If $P(k)$ is true, then $T_k = 4k^2 - 1$ and $T_k = \dfrac{T_{k-1}(2k+1)}{2k-3}$. $P(k + 1)$: $T_{k+1} = \dfrac{T_k\left(2(k+1)+1\right)}{2(k+1)-3}$ $= \dfrac{T_k(2k+3)}{2k-1}$ $= \dfrac{\left(4k^2-1\right)(2k+3)}{2k-1}$ $= \dfrac{8k^3+12k^2-2k-3}{2k-1}$ $= \dfrac{(2k-1)\left(4k^2+8k+3\right)}{2k-1}$ $= 4k^2+8k+3$ $= 4\left(k^2+2k+1\right)-1$ $= 4(k+1)^2-1$ $= T_{k+1}$ As $P(2)$ is true, and $P(k)$ implies $P(k + 1)$, $P(n)$ is true for all $n > 1$.	MEX–P2 Further Proof by Mathematical Induction MEX12–1, 12–2, 12–7, 12–8 Bands E3–E4 • Provides the correct algebraic manipulation of powers to prove $P(k + 1)$. AND • Provides the correct solution . . . . . .3 • Uses $P(k)$ in $P(k + 1)$ to derive $\dfrac{\left(4k^2-1\right)(2k+3)}{2k-1}$ . . . . .2 • Provides the correct proof for $P(2)$ 1
(ii) $\displaystyle\sum_{n=1}^{k}\left(4n^2-1\right) = \left(4\sum_{n=1}^{k} n^2\right) - k$ $= \dfrac{4k(k+1)(2k+1)}{6} - k$ $= \dfrac{\left(4k^3+6k^2+2k\right)-3k}{3}$ $= \dfrac{k\left(4k^2+6k-1\right)}{3}$ $= \dfrac{1}{3}k\left(4k^2+6k-1\right)$ Therefore, the sum of n terms is $\dfrac{1}{3}n\left(4k^2+6k-1\right)$.	MEX–P1 The Nature of Proof MEX12–1, 12–2, 12–7, 12–8 Band E3 • Provides the correct solution . . . . . .2 • Applies properties of summation. . .1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 16	
(a) (i) $\dot{x} = 5,\ x = 5t$ $\dot{y} = 13 - 10t,\ y = 13t - 5t^2$	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Band E3 • Provides the correct solution 1
(ii) Horizontally: $\ddot{x} = -0.5\dot{x}$ $\frac{d\dot{x}}{dt} = -0.5\dot{x}$ $\int_5^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -0.5\int_0^t dt$ $\ln\left\|\frac{\dot{x}}{5}\right\| = -0.5t$ $e^{-\frac{t}{2}} = \frac{\dot{x}}{5}$ $\dot{x} = 5e^{-\frac{t}{2}}$ Vertically: $\ddot{y} = -10 - 0.5\dot{y}$ $\frac{d\dot{y}}{dt} = -10 - 0.5\dot{y}$ $\int_{13}^{\dot{y}} \frac{d\dot{y}}{20 + \dot{y}} = -0.5\int_0^t dt$ $\ln\left\|\frac{20 + \dot{y}}{20 + 13}\right\| = -0.5t$ $e^{-\frac{t}{2}} = \frac{20 + \dot{y}}{33}$ $\dot{y} = 33e^{-\frac{t}{2}} - 20$	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E3–E4 • Provides the correct solution . . . . . . 2 • Provides the correct integration of $\ddot{x}$ OR $\ddot{y}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) The maximum height occurs when $\dot{y} = 0$. $0 = 33e^{-\frac{t}{2}} - 20$ $-\frac{t}{2} = \ln\left(\frac{20}{33}\right)$ $t = 1$ Vertical distance: $\dot{y} = 33e^{-\frac{t}{2}} - 20$ $y = -66e^{-\frac{t}{2}} - 20t + c$ When $t = 0,\ y = 0$. $\therefore c = 66$ $y = -66e^{-\frac{t}{2}} - 20t + 66$ $= -66e^{-\frac{1}{2}} - 20(1) + 66$ $= 5.97$ m Therefore, the rock can be projected to a maximum height of 5.97 m. *Note: Consequential on answer to **Question 16(a)(ii)**.*	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E3–E4 • Provides the correct solution . . . . . . 3 • Provides the correct integration of y . . . . 2 • Provides the correct value of t 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) $v = (k + v_0)a^{bt} - k$ $x = \int v\,dt$ $= \dfrac{(k + v_0)a^{bt}}{b\ln a} - kt + c$ $\because t = 0,\ x = 0$ $\therefore c = -\dfrac{k + v_0}{b\ln a}$ $\therefore x = \dfrac{(k + v_0)a^{bt}}{b\ln a} - kt + \dfrac{k + v_0}{b\ln a}$ $= \dfrac{(k + v_0)(a^{bt} - 1)}{b\ln a} - kt$ $\because v = (k + v_0)a^{bt} - k,\ a^{bt} = \dfrac{v + k}{v_0 + k}$, and $\log_a\left\|\dfrac{v + k}{v_0 + k}\right\| = bt,\ t = \dfrac{\ln\left\|\dfrac{v + k}{v_0 + k}\right\|}{b\ln a}$. $x = \dfrac{(k + v_0)\left(\dfrac{v + k}{v_0 + k} - 1\right)}{b\ln a} - \dfrac{k\ln\left\|\dfrac{v + k}{v_0 + k}\right\|}{b\ln a}$ $= \dfrac{(v + k - k - v_0) - k\ln\left\|\dfrac{v + k}{v_0 + k}\right\|}{b\ln a}$ $= \dfrac{1}{b\ln a}\left(v - v_0 - k\ln\left\|\dfrac{v + k}{v_0 + k}\right\|\right)$	MEX–M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Band E4 • Uses substitution AND algebraic manipulation to provide the correct solution . . . . 3 • Uses algebraic manipulation to obtain any TWO of c, a^{bt} OR t. . 2 • Uses integration to find the value of x in terms of t 1
(c) (i) $(e^{i\theta} + e^{-i\theta})^4 = \binom{4}{0}(e^{i\theta})^4(e^{-i\theta})^0$ $+ \binom{4}{1}(e^{i\theta})^3(e^{-i\theta})^1$ $+ \binom{4}{2}(e^{i\theta})^2(e^{-i\theta})^2$ $+ \binom{4}{3}(e^{i\theta})^1(e^{-i\theta})^3$ $+ \binom{4}{4}(e^{i\theta})^0(e^{-i\theta})^4$ $= e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}$	MEX–N2 Using Complex Numbers MEX12–4, 12–7 Band E3 • Provides the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\left(e^{i\theta}+e^{-i\theta}\right)^n=\sum_{r=0}^{n}\binom{n}{r}\left(e^{i\theta}\right)^{n-r}\left(e^{-i\theta}\right)^r$ $2\left(e^{i\theta}+e^{-i\theta}\right)^n=\left[\binom{n}{0}\left(e^{i\theta}\right)^n\left(e^{-i\theta}\right)^r\right.$ $+\binom{n}{1}\left(e^{i\theta}\right)^{n-1}\left(e^{-i\theta}\right)^1$ $+\binom{n}{2}\left(e^{i\theta}\right)^{n-2}\left(e^{-i\theta}\right)^2+\ldots$ $\left.+\binom{n}{n}\left(e^{i\theta}\right)^0\left(e^{-i\theta}\right)^n\right]$ $+\left[\binom{n}{n}\left(e^{i\theta}\right)^0\left(e^{-i\theta}\right)^n\right.$ $+\binom{n}{n-1}\left(e^{i\theta}\right)^1\left(e^{-i\theta}\right)^{n-1}$ $+\binom{n}{n-2}\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right)^{n-2}$ $\left.+\ldots+\binom{n}{0}\left(e^{i\theta}\right)^n\left(e^{-i\theta}\right)^0\right]$ $\because\binom{n}{r}=\binom{n}{n-r},\ \binom{n}{r}+\binom{n}{n-r}=2\binom{n}{r}$ $2\left(e^{i\theta}+e^{-i\theta}\right)^n=\sum_{r=0}^{n}2\binom{n}{r}\left(e^{(n-r)i\theta}e^{-ri\theta}+e^{(r)i\theta}e^{-(n-r)i\theta}\right)$ $=2\sum_{r=0}^{n}\binom{n}{r}\left(e^{(n-2r)i\theta}+e^{-(n-2r)i\theta}\right)$ $=2\sum_{r=0}^{n}\binom{n}{r}2\cos((n-2r)\theta)$ $\left(e^{i\theta}+e^{-i\theta}\right)^n=\sum_{r=0}^{n}\binom{n}{r}2\cos((n-2r)\theta)$ $=2\sum_{r=0}^{n}\binom{n}{r}\cos((n-2r)\theta)$	MEX–N2 Using Complex Numbers MEX–P1 The Nature of Proof MEX12–1, 12–2, 12–4, 12–7, 12–8 Band E4 • Uses the conjugate property to provide the correct solution . . . . . 3 • Uses the symmetry of binomial coefficients to derive $2\sum_{r=0}^{n}\binom{n}{r}\left(e^{(n-2r)i\theta}+e^{-(n-2r)i\theta}\right)$. OR • Equivalent merit . . . . 2 • Adds TWO series AND combines the terms 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) $\int \left(e^{i\theta}+e^{-i\theta}\right)^6 d\theta = 2\int \binom{6}{0}\cos 6\theta + \binom{6}{1}\cos 4\theta$ $+\binom{6}{2}\cos 2\theta + \binom{6}{3}\cos\theta$ $+\binom{6}{4}\cos(-2\theta)$ $+\binom{6}{5}\cos(-4\theta)$ $+\binom{6}{6}\cos(-6\theta)$ (from part (c)(ii)) $= 2\int \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta$ $+20\cos 0° + 15\cos(-2\theta)$ $+6\cos(-4\theta)$ $+\cos(-6\theta)d\theta$ $\because \cos(-\theta)=\cos\theta$ $\int \left(e^{i\theta}+e^{-i\theta}\right)^6 d\theta = 2\int 2\cos 6\theta + 12\cos 4\theta$ $+30\cos 2\theta + 40 d\theta$ $= \frac{2\sin 6\theta}{3} + 6\sin 4\theta$ $+30\sin 2\theta + 80\theta + C$ $= \frac{2\sin 6\theta}{3} + 6\sin 4\theta$ $+30\sin 2\theta + 80\theta + C$ *Note: Consequential on answer to **Question 16(c)(ii).***	MEX–C1 Further Integration MEX12–1, 12–4, 12–5, 12–7 Bands E3–E4 • Provides the correct solution . . . . . .2 • Uses part (c)(ii) to obtain the sum of cosines 1