CSSA

2023

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 2

Morning Session
Monday, 7 August 2023

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–13)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10

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1 A composite number is a positive integer that can be formed by multiplying two smaller integers, both larger than 1. Which set of positive integers n is this statement true for?

"The number $n^2 - 1$ is composite."

A. $n \geq 1$

B. $n \geq 2$

C. $n \geq 3$

D. $n \geq 4$

2 Consider the following two statements.

Statement X: The sum of two rational numbers is rational.

Statement Y: The sum of two irrational numbers is irrational.

Which of the statements is true?

A. X and Y are both true.

B. X is true and Y is false.

C. X is false and Y is true.

D. X and Y are both false.

3 Maeve is asked to prove or disprove the following statement for x and y positive real numbers: $\frac{1}{x^2}+\frac{1}{y^2} \geq \frac{2}{xy}$. Her working is shown below.

$$\text{We know: } \frac{1}{(x-y)^2} \geq 0 \Leftrightarrow \frac{1}{x^2-2xy+y^2} \geq 0$$

$$\Leftrightarrow \frac{1}{x^2+y^2} \geq \frac{1}{2xy} \qquad \textbf{line A}$$

$$\Leftrightarrow x^2+y^2 \leq 2xy \qquad \textbf{line B}$$

$$\Leftrightarrow \frac{x^2+y^2}{x^2y^2} \leq \frac{2xy}{x^2y^2} \qquad \textbf{line C}$$

$$\Leftrightarrow \frac{1}{x^2}+\frac{1}{y^2} \leq \frac{2}{xy} \qquad \textbf{line D}$$

So the statement is false.

On which line of working has Maeve made a mistake?

A. line A

B. line B

C. line C

D. line D

4 The amount of apples, bananas and oranges sold by a fruit seller over a year is shown in the table below.

Fruit	Amount Sold (tonnes)	Profit ($/tonne)
Apples	25	530
Bananas	55	380
Oranges	10	410

Let $\underset{\sim}{a} = \begin{pmatrix}25\\55\\10\end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix}530\\380\\410\end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix}1\\1\\1\end{pmatrix}$. Which of the following expressions calculates the average profit in dollars per tonne of fruit sold over the year?

A. $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{\underset{\sim}{b} \cdot \underset{\sim}{c}}$

B. $\dfrac{\underset{\sim}{b} \cdot \underset{\sim}{c}}{\underset{\sim}{a} \cdot \underset{\sim}{c}}$

C. $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{c}}$

D. $\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{a} \cdot \underset{\sim}{c}}$

5 Which of the following expressions is equivalent to $\int \ln(x^2+1)\,dx$?

A. $x\ln(x^2+1) - 2x + 2\tan^{-1}x + c$

B. $x\ln(x^2+1) - \ln(x^2+1) + c$

C. $\ln(x^2+1) - 2x + 2\tan^{-1}x + c$

D. $\ln(x^2+1) - x\ln(x^2+1) + c$

6 Consider the complex, non-real cube roots of unity ω and ω^2. What is the value of $(1-\omega+\omega^2)(1+\omega-\omega^2)$?

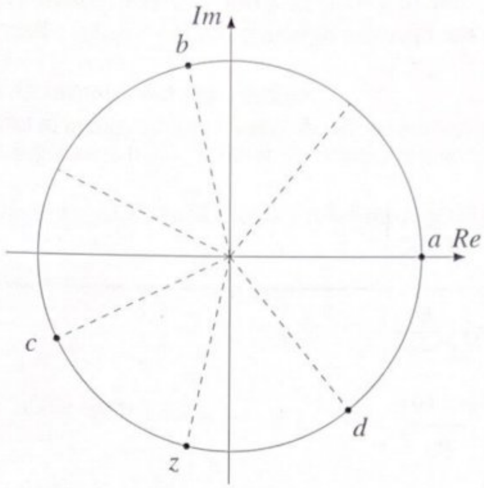
A. 0

B. 1

C. 2

D. 4

7 The complex numbers a, b, c and d and z are solutions to $z^7 = 1$ as shown in the Argand diagram below.



Which of the following is a cube root of z?

A. a

B. b

C. c

D. d

8 Consider the transitive property of implication:

$$(a \Rightarrow b \text{ AND } b \Rightarrow c) \Rightarrow (a \Rightarrow c).$$

Which of the following is the contrapositive of the transitive property of implication?

A. $\neg(a \Rightarrow c) \Rightarrow (\neg(a \Rightarrow b) \text{ OR } \neg(b \Rightarrow c))$

B. $\neg(a \Rightarrow c) \Rightarrow (\neg(a \Rightarrow b) \text{ AND } \neg(b \Rightarrow c))$

C. $\neg(a \Rightarrow c) \Rightarrow \neg(\neg(a \Rightarrow b) \text{ OR } \neg(b \Rightarrow c))$

D. $\neg(a \Rightarrow c) \Rightarrow \neg(\neg(a \Rightarrow b) \text{ AND } \neg(b \Rightarrow c))$

9 An object of unit mass falls from rest with downward velocity of $v\,\mathrm{ms}^{-1}$. The object undergoes acceleration due to gravity of $g\,\mathrm{ms}^{-2}$. It also experiences air resistance of magnitude $kv\,\mathrm{ms}^{-2}$, in the opposite direction to the velocity, where k is some positive constant.

Which is the correct expression for the distance x m it has fallen in terms of its downward velocity $v\,\mathrm{ms}^{-1}$?

A. $x = -\frac{v}{k} - \frac{g}{k^2}\ln\left|\frac{g}{g-kv}\right|$

B. $x = -\frac{v}{k} + \frac{g}{k^2}\ln\left|\frac{g}{g-kv}\right|$

C. $x = \frac{v}{k} - \frac{g}{k^2}\ln\left|\frac{g-kv}{g}\right|$

D. $x = \frac{v}{k} + \frac{g}{k^2}\ln\left|\frac{g-kv}{g}\right|$

10 The function $F(x)$ is a primitive of $f(x)$, that is $F'(x) = f(x)$. Which of the following is true?

A. $\int\left(\frac{d}{dx}\int_a^b f(x)dx\right)dx = f(b) - f(a)$

B. $\int_a^b\left(\frac{d}{dx}\int f(x)dx\right)dx = f(b) - f(a)$

C. $\frac{d}{dx}\int\left(\int_a^b f(x)dx\right)dx = F(b) - F(a)$

D. $\frac{d}{dx}\int_a^b\left(\int f(x)dx\right)dx = F(b) - F(a)$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

Your responses for Questions 11–16 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Express $2\sqrt{2}e^{-\frac{3\pi}{4}i}$ in the form $x+iy$. **2**

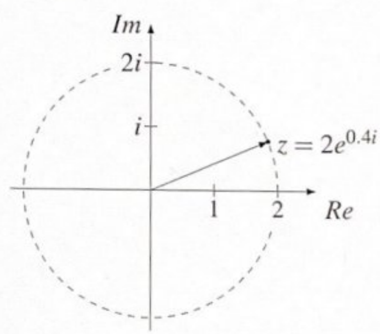
(b) Consider the two points $A(2,2,2)$ and $B(2,-2,2)$.

(i) Find $\overrightarrow{AB}$. **1**

(ii) Find $|\overrightarrow{AB}|$. **1**

(iii) Find $\angle AOB$. Give your answer correct to the nearest degree. **2**

(c) Consider the complex number $z = 2e^{0.4i}$, as sketched below. **3**



Copy and clearly label this Argand diagram, and sketch the four points represented by z, $\overline{z}$, $-\overline{z}$, and $z-\overline{z}$.

(d) A particle starts at rest from the origin and moves in a straight line such that its displacement x metres at time t seconds is determined by the equation $\ddot{x} = -16x$. How long will it take for the particle to first return to the origin? **3**

(e) Show for any complex numbers z and w, that $\overline{zw} = \overline{z} \times \overline{w}$. **3**

Question 12 (15 marks)

(a) Consider the two lines

$$L_1 : \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \quad \text{and} \quad L_2 : \begin{pmatrix} -1 \\ 2 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}.$$

(i) Find the value of k given $B\,(9, k, 24)$ lies on L_2. **2**

(ii) Find the point of intersection of L_1 and L_2. **3**

(b) Find $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}}\,dx$. **4**

(c) Solve $z^2 + (7-i)z + 16 + 4i = 0$. **3**

(d) Find $\displaystyle\int \frac{x^3-2}{x^3+x}\,dx$. **3**

Question 13 (15 marks)

(a) Prove that $\left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right) \geq 4$ where $a, b > 0$. **2**

(b) The graph of a polynomial function $f(x) = (x+3)(x-2)(x^2+bx+c)$ has a y-intercept of -6 and passes through the point $(1, -4)$.

(i) Find the two complex roots of the equation $f(x) = 0$. **2**

(ii) Express these two complex roots in the form $r(\cos\theta + i\sin\theta)$. **2**

(iii) Plot all four solutions to $f(x) = 0$ on an Argand diagram. **2**

(iv) Write down the name of the quadrilateral formed by these four points. **1**

(c) Consider the sphere with vector equation $\left| \underset{\sim}{r} - \begin{pmatrix} 3 \\ -12 \\ 4 \end{pmatrix} \right| = 3$.

(i) Show the point $(5, -10, 3)$ lies on the sphere. **1**

(ii) Find the point on the sphere farthest from the origin. **2**

(d) Prove by mathematical induction that $3n^2 - 3n \leq 2^n - 1$ for $n \geq 7$. **3**

Question 14 (15 marks)

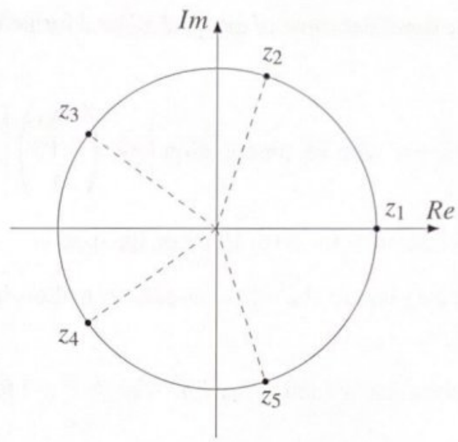
(a) The number z is a fifth root of unity where $z \neq 1$, that is $z^5 = 1$.

(i) Show that $(z+z^{-1})^2 + (z+z^{-1}) - 1 = 0$. **2**

(ii) If $z = e^{i\theta}$, show $\cos\theta = \dfrac{z+z^{-1}}{2}$. **1**

(iii) Hence show $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$. **2**

(iv) Consider the five fifth roots of unity z_1, z_2, z_3, z_4 and z_5 as shown in the diagram below. **3**



Show that $\left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| = \dfrac{1+\sqrt{5}}{2}$.

(b) The position of an object after t seconds is given by the vector equation

$$\underset{\sim}{r} = \cos\frac{\pi t}{4}\underset{\sim}{i} + \left(\cos\frac{\pi t}{4} + \sin\frac{\pi t}{4}\right)\underset{\sim}{j} + \sin\frac{\pi t}{4}\underset{\sim}{k}.$$

(i) What is the position of the object after 3 seconds? **1**

(ii) Find the vector equation of the tangent to the path taken by the object after 3 seconds. **3**

(c) Evaluate $\displaystyle\int_0^9 \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$. **3**

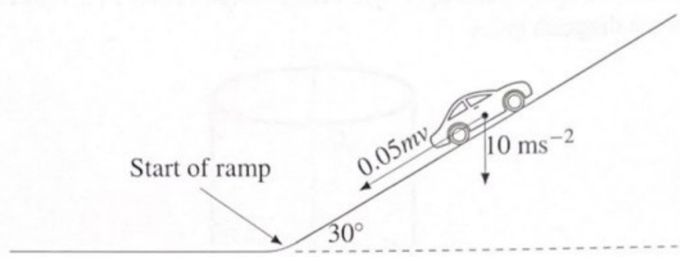
Question 15 (15 marks)

(a) Consider the function $f(x) = e^{-x}\cos x$, with domain $x \in \left[0, \dfrac{3\pi}{2}\right]$. **4**

Show that the ratio of the area above the x-axis to the area below the x-axis is

$$\frac{e^{\pi}\left(e^{\frac{\pi}{2}}+1\right)}{e^{\pi}+1}.$$

(b) An emergency ramp is a steep ramp designed for cars on a highway to safely stop if their brakes have failed. A car rolls onto an emergency ramp inclined at 30° to the horizontal at 144 km/h without its brakes on as shown in the diagram below. **4**



The car experiences resistance due to friction of magnitude $0.05mv$ newtons where m kg is the car's mass and v ms^{-1} is the car's velocity at time t seconds after entering the ramp.

Assuming the acceleration due to gravity is $10\ \text{ms}^{-2}$, calculate how far the car travels up the ramp before stopping. Give your answer correct to the nearest metre.

(c) Let $I_n = \displaystyle\int \frac{\cos nx}{\sin x}\,dx$.

(i) Show that $\cos\left((n-2)x\right) - \cos nx = 2\sin\left((n-1)x\right)\sin x$. **2**

(ii) Show that $I_n - I_{n-2} = \dfrac{2\cos\left((n-1)x\right)}{n-1} + c$. **2**

(iii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\cos 2x - \cos 6x}{\sin x}\,dx$. **3**

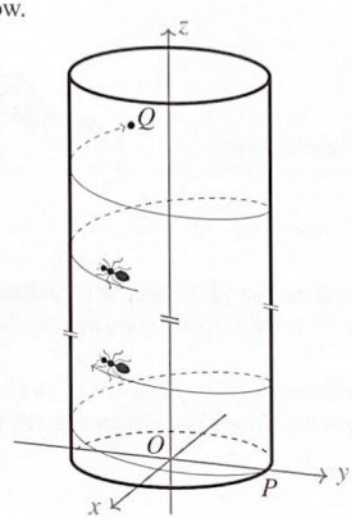
Question 16 (15 marks)

(a) Consider the tile below consisting of three equilateral triangles of side length 1 unit. **3**

Prove the following result for positive integers n, using mathematical induction:

An equilateral triangle of side length 2^n units may be covered by the tiles above (in any orientation) such that a single equilateral triangle of side length 1 unit is left over at one of the vertices. The tiles may not overlap.

(b) An ant follows a spiral path up a cylindrical column from $P(0,7,0)$ to the point Q as shown in the diagram below.



The ant's position $\underset{\sim}{r}$ in centimetres after t seconds is given by the vector equation below.

$$\underset{\sim}{r}(t) = 7\sin\frac{\pi t}{32}\underset{\sim}{i} + 7\cos\frac{\pi t}{32}\underset{\sim}{j} + \frac{t}{15}\underset{\sim}{k}$$

(i) Find the coordinates of the point Q if $\left|\overrightarrow{OQ}\right| = 25$. **3**

(ii) How many times has the ant crossed the line $x = 7$ on its journey to Q? **2**

(iii) The ant crawls back from Q to P along the shortest path possible. How far did it crawl on this leg of its journey? Give your answer correct to one decimal place. **2**

Question 16 continues on page 13

(c) A projectile of unit mass is launched vertically upwards from a horizontal plane with initial speed v_0 ms^{-1}. The projectile experiences a resistive force that causes an acceleration with a magnitude of $0.01v^2$ ms^{-2}, where v ms^{-1} is the velocity of the projectile. The acceleration due to gravity is 10 ms^{-2}. **5**

The projectile lands on the horizontal plane at $\frac{1}{7}$ the speed that it was launched.

Find the value of v_0, correct to 1 decimal place.

End of Examination

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EXAMINERS

David Houghton (Convenor)	Oxley College, Burradoo
Geoff Carroll	Sydney Grammar School, Darlinghurst
Stephen Ewington	Ascham School, Edgecliff
Rebekah Johnson	Loreto Kirribilli, Kirribilli
Svetlana Onisczenko	Meriden School, Strathfield

Questions 7, 11(c), 14(a)(iv), 15(b), and 16(b) – Diagrams created by the 2023 CSSA Trial HSC Examinations Mathematics Extension 2 Committee