CSSA

2023

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MARKING GUIDELINES

Mathematics Extension 2

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	C	MEX12-2	E1-E2
2	B	MEX12-2	E1-E2
3	A	MEX12-2	E1
4	D	MEX12-3	E2
5	A	MEX12-5	E2-E3
6	D	MEX12-4	E2
7	C	MEX12-4	E3-E4
8	A	MEX12-1	E3-E4
9	B	MEX12-6	E3-E4
10	C	MEX12-5	E4

Question 1 (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E1-E2*

Solution	Mark
$n^2-1=(n+1)(n-1)$. If $n+1\geq 2$, and $n-1\geq 2$, the definition is satisfied. Therefore $n\geq 3$ satisfies the definition. Hence C	1

Question 2 (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E1-E2*

Solution	Mark
The sum of two rational numbers is rational, hence statement X is true. However, the sum of two irrational numbers could be either rational or irrational. e.g. $(1+\sqrt{2})+(1-\sqrt{2})=2\in\mathbb{Q}$ whereas $\sqrt{2}+\sqrt{3}\notin\mathbb{Q}$. Therefore Statement Y is false. Hence B	1

Question 3 (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E1*

Solution	Mark
$\frac{1}{x^2-2xy+y^2} \neq \frac{1}{x^2+y^2} - \frac{1}{2xy}$, hence the error is at line A. Hence A	1

Question 4 (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Solution	Mark
$\frac{\$\text{profit}}{\text{tonnes sold}} = \frac{25\times 530 + 55\times 380 + 10\times 410}{25+55+10}$ $= \frac{\begin{pmatrix}25\\55\\10\end{pmatrix}\cdot\begin{pmatrix}530\\380\\410\end{pmatrix}}{\begin{pmatrix}25\\55\\10\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{a}\cdot\underset{\sim}{c}}$ Hence D	1

Question 5 (1 mark)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Solution	Mark
$I = \int \ln(x^2+1)\,dx$ Let $u = \ln(x^2+1)$, $v = x$ So $du = \frac{2x}{x^2+1}$, $dv = dx$ $I = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx$ $= x\ln(x^2+1) - 2\int 1 - \frac{1}{x^2+1}\,dx$ $= x\ln(x^2+1) - 2x + 2\tan^{-1}x + c$ Hence A	1

Question 6 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Solution	Mark
$$\begin{aligned}\omega^3-1&=0\\(\omega-1)\left(\omega^2+\omega+1\right)&=0,\text{ and since }\omega\notin\mathbb{R}\\\omega^2&=-(\omega+1)\\\text{Therefore }\left(1-\omega+\omega^2\right)\left(1+\omega-\omega^2\right)&=(1-\omega-\omega-1)\left(-\omega^2-\omega^2\right)\\&=-2\omega\times-2\omega^2\\&=4\omega^3=4,\text{ since }\omega^3=1.\end{aligned}$$ Hence D	1

Question 7 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E3-E4*

Solution	Mark
$\arg z=-\frac{4\pi}{7}$ $\arg a=0$, hence $\arg\left(a^3\right)=0\neq\arg z$. $\arg b=\frac{4\pi}{7}$, hence $\arg\left(b^3\right)=\frac{12\pi}{7}$, equivalent to $-\frac{2\pi}{7}\neq\arg z$. $\arg c=-\frac{6\pi}{7}$, hence $\arg\left(c^3\right)=-\frac{18\pi}{7}$, equivalent to $-\frac{4\pi}{7}=\arg z$. $\arg d=-\frac{2\pi}{7}$, hence $\arg\left(d^3\right)=-\frac{6\pi}{7}\neq\arg z$. Hence C	1

Question 8 (1 mark)
Outcomes Assessed: *MEX12-1*
Targeted Performance Bands: *E3-E4*

Solution	Mark
Contrapositive is $\neg(a\Rightarrow c)\Rightarrow\neg(a\Rightarrow b\text{ AND }b\Rightarrow c)$. Now, $\neg(p\text{ AND }q)$ is logically equivalent to $\neg p\text{ OR }\neg q$, hence $\neg(a\Rightarrow c)\Rightarrow\left(\neg(a\Rightarrow b)\text{ OR }\neg(b\Rightarrow c)\right)$. Hence A	1

Question 9 (1 mark)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
With downwards being positive, the acceleration due to resistance kv acts upwards, so: $\dot{v} = g - kv$ $v\dfrac{dv}{dx} = g - kv$ $\dfrac{dx}{dv} = \dfrac{v}{g-kv}$ $= -\dfrac{1}{k} \times \dfrac{-kv}{g-kv}$ $= -\dfrac{1}{k}\left(\dfrac{g-kv}{g-kv} - \dfrac{g}{g-kv}\right)$ $\dfrac{dx}{dv} = -\dfrac{1}{k} - \dfrac{g}{k^2} \times \dfrac{-k}{g-kv}$ $x = -\dfrac{v}{k} - \dfrac{g}{k^2}\ln\lvert g-kv\rvert + c$ Now, when $x = 0,\ v = 0$, so $0 = 0 - \dfrac{g}{k^2}\ln g + c$ $x = -\dfrac{v}{k} - \dfrac{g}{k^2}\ln\lvert g-kv\rvert + \dfrac{g}{k^2}\ln g$ $x = -\dfrac{v}{k} + \dfrac{g}{k^2}\ln\left\lvert\dfrac{g}{g-kv}\right\rvert$ Hence B	1

Question 10 (1 mark)
Outcomes Assessed: MEX12-5
Targeted Performance Bands: E4

Solution	Mark
$\text{LHS of A} = \displaystyle\int\left(\frac{d}{dx}\int_a^b f(x)dx\right)dx = \int\left(\frac{d}{dx}\Big[F(b)-F(a)\Big]\right)dx$ $= \displaystyle\int\Big(0\Big)dx = c \neq \text{ RHS}$ $\text{LHS of B} = \displaystyle\int_a^b\left(\frac{d}{dx}\int f(x)dx\right)dx = \int_a^b\left(\frac{d}{dx}(F(x)+c)\right)dx$ $= \displaystyle\int_a^b\Big(f(x)\Big)dx = F(b)-F(a) \neq \text{ RHS}$ $\text{LHS of C} = \displaystyle\frac{d}{dx}\int\left(\int_a^b f(x)dx\right)dx = \frac{d}{dx}\int\Big(F(b)-F(a)\Big)dx$ $= \dfrac{d}{dx}\Big[xF(b) - xF(a) + c\Big] = F(b) - F(a) = \text{RHS}$ $\text{LHS of D} = \displaystyle\frac{d}{dx}\int_a^b\left(\int f(x)dx\right)dx = \frac{d}{dx}\int_a^b\Big(F(x)+c\Big)dx$ $= \dfrac{d}{dx}\Big[\text{some real number}\Big] = 0 \neq \text{ RHS}$ Hence C	1

Section II

90 marks

Question 11 (15 marks)

Question 11(a) (2 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1*

Criteria	Marks
• correct solution	2
• progress towards correct solution	1

Sample Answer:

$$2\sqrt{2}e^{-\frac{3\pi}{4}i} = 2\sqrt{2}\left(-\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}\right) = -2-2i$$

Question 11(b) (i) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E1*

Criteria	Mark
• correct solution	1

Sample Answer:

$$\overrightarrow{AB} = 0\underset{\sim}{i} - 4\underset{\sim}{j} + 0\underset{\sim}{k}$$

Question 11(b) (ii) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E1*

Criteria	Mark
• correct solution	1

Sample Answer:

$$|\overrightarrow{AB}| = \sqrt{4^2} = 4$$

Question 11(b) (iii) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution with justification (dot product or cosine rule)	2
• progress towards correct solution	1

Sample Answer:

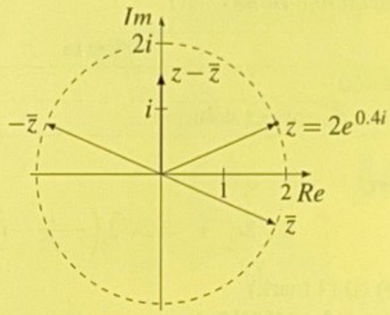
$$\cos\angle AOB = \frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}|\times|\overrightarrow{OB}|} = \frac{4-4+4}{\left(\sqrt{2^2+2^2+2^2}\right)^2} = \frac{1}{3}.\ \text{Hence } \angle AOB = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71^\circ.$$

Question 11(c) (3 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Criteria	Marks
• correct sketch including the point $z-\bar{z}$, not just the vector from $\bar{z}$ to z	3
• two correct points, probably z and $\bar{z}$ as reflections over x-axis	2
• at least one correct point, representing modulus and argument	1

Sample Answer:

The image to the right displays the four points. Note that the question is not asking for the vector from $\bar{z}$ to z, but rather the number $z-\bar{z}$, which sits between i and $2i$.



Question 11(d) (3 marks)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	3
• evaluates period	2
• evaluates n	1

Sample Answer:

The particle is moving in simple harmonic motion, so $a=-4^2x \;\therefore\; n=4$.
So period $=\frac{2\pi}{4}=\frac{\pi}{2}$. The particle returns to the origin in half a period, hence in $\frac{\pi}{4}$ s.

Question 11(e) (3 marks)
Outcomes Assessed: *MEX12-1, MEX12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• equates LHS and RHS to complete the proof	3
• expands LHS or RHS correctly	2
• correctly sets up real and imaginary parts of both numbers and substitutes	1

Sample Answer:

Let $z=a+ib$ and $w=c+id$ where $a,b,c,d\in\mathbb{R}$.

$$\begin{aligned}\text{LHS}&=\overline{zw}\\&=\overline{(a+ib)(c+id)}\\&=\overline{(ac-bd)+i(ad+bc)}\\&=(ac-bd)-i(ad+bc)\end{aligned}$$

$$\begin{aligned}\text{RHS}&=\bar{z}\times\bar{w}\\&=(a-ib)(c-id)\\&=(ac-bd)-i(ad+bc)=\text{LHS}\end{aligned}$$

Question 12 (15 marks)

Question 12(a) (i) (2 marks)

Outcomes Assessed: *MEX12-3*

Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	2
• correctly substitutes in at least one dimension	1

Sample Answer:

$$\begin{pmatrix} 9 \\ k \\ 24 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

So, $9 = -1 + 2\mu \quad \Rightarrow \quad \mu = 5.$
Further, $k = 2 - 3\mu \quad \Rightarrow \quad k = -13.$

Question 12(a) (ii) (3 marks)

Outcomes Assessed: *MEX12-3*

Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	3
• finds λ in terms of μ or similar progress	2
• correctly substitutes in at least one dimension	1

Sample Answer:

From $\underset{\sim}{i}$, $7 + \lambda = -1 + 2\mu \quad \Rightarrow \quad \lambda = -8 + 2\mu,$ and from $\underset{\sim}{j}$, $-1 + 3\lambda = 2 - 3\mu$

Substituting the first into the second gives: $-1 + 3(-8 + 2\mu) = 2 - 3\mu$

$$9\mu = 2 + 24 + 1$$

Which gives $\mu = 3$ and $\lambda = -2$. Substituting in L_1 or L_2 gives the point $(5, -7, 12)$.

Question 12(b) (4 marks)

Outcomes Assessed: *MEX12-5*

Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	4
• correctly integrates and correctly substitutes limits	3
• correctly integrates or correctly substitutes limits	2
• devises appropriate substitution or similar progress	1

Sample Answer:

If $x = \sin\theta$ then $dx = \cos\theta \, d\theta$.
When $x = 0$, $\theta = 0$, when $x = \frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{4}$.

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}} \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin\theta \left(1 - \cos^2\theta\right) d\theta$$

$$= \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{3}\left(\tfrac{1}{\sqrt{2}}\right)^3\right) - \left(-1 + \tfrac{1}{3}\right)$$

$$= \left(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{12}\right) + \tfrac{2}{3}$$

$$= \frac{8 - 5\sqrt{2}}{12}$$

Question 12(c) (3 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	3
• finds square root of discriminant	2
• finds discriminant or attempts quadratic formula	1

Sample Answer:

Let a and b be real numbers such that $(a+ib)^2 = \Delta = (7-i)^2 - 4(16+4i) = -16-30i.$
Hence $a^2 - b^2 = -16$ while $2ab = -30$. By inspection, $a = 3$ and $b = -5$.

Therefore $z = \dfrac{-(7-i) \pm (3-5i)}{2} = \dfrac{-4-4i}{2}$ or $\dfrac{-10+6i}{2} = -2-2i$ or $-5+3i$.

Question 12(d) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	3
• finds correct partial fraction expression to integrate	2
• attempts partial fractions or similar progress	1

Sample Answer:

$$\int \frac{x^3-2}{x^3+x}\,dx = \int \left(\frac{x^3+x}{x^3+x} - \frac{x+2}{x(x^2+1)}\right) dx$$

Now, if $\dfrac{x+2}{x(x^2+1)} \equiv \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

Then $(A+B)x^2 + Cx + A \equiv x+2$
Hence $A = 2,\, B = -2,\, C = 1.$

And further, $$\int \frac{x^3-2}{x^3+x}\,dx = \int \left(1 - \frac{2}{x} + \frac{2x-1}{x^2+1}\right) dx$$
$$= x - 2\ln x + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$
$$= x - 2\ln|x| + \ln|x^2+1| - \tan^{-1} x + c$$

Question 13 (15 marks)

Question 13(a) (2 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• completed proof	2
• significant progress towards proof	1

Sample Answer:

$a, b > 0 \quad \Rightarrow \quad \left(\sqrt{ab} - \frac{1}{\sqrt{ab}}\right)^2 \geq 0$

$\Rightarrow \quad ab - 2 + \frac{1}{ab} \geq 0$

$\Rightarrow \quad ab + 2 + \frac{1}{ab} \geq 4$

$\Rightarrow \quad \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) \geq 4$

Question 13(b) (i) (2 marks)
***Outcomes Assessed:* MEX12-4**
***Targeted Performance Bands:* E2**

Criteria	Marks
• correct solution	2
• evaluates b or c or work of equivalent merit	1

Sample Answer:

$f(x) = (x+3)(x-2)(x^2+bx+c)$
$f(0) = -6 = 3\times -2\times c \quad \Rightarrow \quad c = 1$
$f(1) = -4 = 4\times -1\times(1+b+1) \quad \Rightarrow \quad b+2 = 1 \quad \Rightarrow \quad b = -1$
Now the two non-real roots of the equation are given by:
$x^2 - x + 1 = 0$
$x^2 - x + \frac{1}{4} = -\frac{3}{4}$
$\left(x-\frac{1}{2}\right)^2 = \left(i\frac{\sqrt{3}}{2}\right)^2$
$x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Question 13(b) (ii) (2 marks)
***Outcomes Assessed:* MEX12-4**
***Targeted Performance Bands:* E2**

Criteria	Marks
• correct solution	2
• one correct root or both correct moduluses or both correct arguments	1

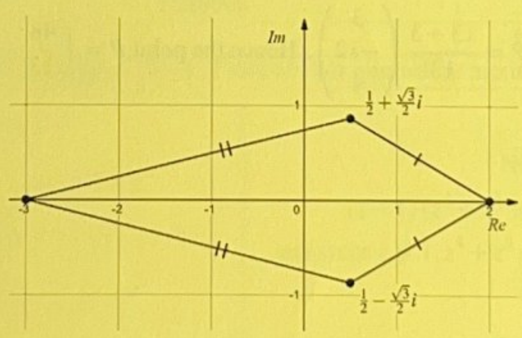
Sample Answer:

$x = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or $1\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

Question 13(b) (iii) (2 marks)
***Outcomes Assessed:* MEX12-4**
***Targeted Performance Bands:* E2-E3**

Criteria	Marks
• correct solution	2
• at least two correct points	1

Sample Answer:



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Question 13(b) (iv) (1 mark)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E1

Criteria	Mark
• names the kite	1

Sample Answer:

Two pairs of adjacent sides are equal, hence they form a kite.

Question 13(c) (i) (1 mark)
Outcomes Assessed: MEX12-3
Targeted Performance Bands: E2

Criteria	Mark
• correct solution	1

Sample Answer:

$$\text{LHS}=\left|\begin{pmatrix}5\\-10\\3\end{pmatrix}-\begin{pmatrix}3\\-12\\4\end{pmatrix}\right|=\sqrt{2^2+2^2+(-1)^2}=3=\text{RHS}$$

Question 13(c) (ii) (2 marks)
Outcomes Assessed: MEX12-3, MEX12-7
Targeted Performance Bands: E3

Criteria	Marks
• correct solution	2
• some progress towards correct solution	1

Sample Answer:

Centre of sphere is $(3,-12,4)$, and the point on the sphere farthest from the origin would be 3 units (one radius) farther along the vector $\overrightarrow{OC}$.
Now, $|\overrightarrow{OC}|=\sqrt{3^2+(-12)^2+4^2}=13$, so the point P on the surface of the sphere farthest from O will be given by: $\overrightarrow{OP}=\dfrac{13+3}{13}\begin{pmatrix}3\\-12\\4\end{pmatrix}$. Hence the point $P=\left(\dfrac{48}{13},-\dfrac{192}{13},\dfrac{64}{13}\right)$.

Question 13(d) (3 marks)
Outcomes Assessed: *MEX12-2, MEX12-8*
Targeted Performance Bands: *E3*

Criteria	Marks
• Completes the proof correctly	3
• Substitutes the correct expression for $n = k$ and $n = k+1$	2
• Demonstrates the result true for $n = 7$	1

Sample Answer:

RTP: $3n^2 - 3n \leq 2^n - 1$ for $n \geq 7$
PROOF: if $n = 7$, LHS $= 21 \times 6 = 126$
RHS $= 2^7 - 1 = 127 \geq$ LHS. Hence the result is true for $n = 7$.

Let's assume the result is true for some integer k.
That is, assume $3k(k-1) \leq 2^k - 1$ and then attempt to prove that $3(k+1)k \leq 2^{k+1} - 1$.
So, IF $3k(k-1) \leq 2^k - 1$
THEN $6k(k-1) \leq 2^{k+1} - 2$
THEN $6k^2 - 6k + 1 \leq 2^{k+1} - 1$
THEN $3k^2 + 3k + \left(3k^2 - 9k + 1\right) \leq 2^{k+1} - 1$

Now to arrive at the result, we now need $3k^2 - 9k + 1 \geq 0$, which is true for $k \geq \frac{9+\sqrt{81-12}}{6}$.
Further $\frac{9+\sqrt{69}}{6} \approx 2.9$, so for the values of $k \geq 7$ in this result, $3k^2 - 9k + 1 > 0$.

Putting this together, $3(k+1)k < 3(k+1)k + \left(3k^2 - 9k + 1\right) \leq 2^{k+1} - 1$
Therefore, $3(k+1)^2 - 3(k+1) \leq 2^{k+1} - 1$.
Hence if the result is true for $n = k$, the result will be true for $n = k+1$.
By the principle of Mathematical Induction, the result is true for $n \geq 7$.

Question 14 (15 marks)

Question 14(a) (i) (2 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• expands correctly or recognises sum of roots= 0 or equivalent merit	1

Sample Answer:

$$\text{LHS} = (z+z^{-1})^2 + (z+z^{-1}) - 1$$
$$= \frac{1}{z^2}\left(z^4 + 2z^2 + 1 + z^3 + z - z^2\right)$$
$$= \frac{1}{z^2}\left(z^4 + z^3 + z^2 + z + 1\right)$$

Now given $z^5 = 1$
$$(z-1)\left(z^4 + z^3 + z^2 + z + 1\right) = 0$$
and since $z \neq 1, z^4 + z^3 + z^2 + z + 1 = 0$
LHS $= 0 =$ RHS

Question 14(a) (ii) (1 mark)
***Outcomes Assessed:* MEX12-4**
***Targeted Performance Bands:* E2**

Criteria	Mark
• correct solution	1

Sample Answer:

$\text{RHS} = \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta))$, and since cos is even fn and sin is odd
$= \frac{1}{2}(\cos\theta + i\sin\theta + \cos(\theta) - i\sin(\theta)) = \frac{1}{2}\times 2\cos\theta = \text{LHS}$

Question 14(a) (iii) (2 marks)
***Outcomes Assessed:* MEX12-4**
***Targeted Performance Bands:* E3**

Criteria	Marks
• correct solution	2
• some progress towards correct solution	1

Sample Answer:

z is a non-real fifth root of unity so
$z = \text{cis}\,\theta$ where $\theta = \pm\frac{2\pi}{5}$ or $\pm\frac{4\pi}{5}$
From (i) and (ii):
$(2\cos\theta)^2 + (2\cos\theta) - 1 = 0$
$4\cos^2\theta + 2\cos\theta - 1 = 0$

$\cos\theta = \dfrac{-2\pm\sqrt{4+16}}{8} = \dfrac{-1\pm\sqrt{5}}{4}$
Now, $\cos\frac{2\pi}{5} > 0$.
Therefore $\cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$.

Question 14(a) (iv) (3 marks)
***Outcomes Assessed:* MEX12-4**
***Targeted Performance Bands:* E3-E4**

Criteria	Marks
• correct solution	3
• evaluates $\lvert z_2 - z_1\rvert$ and $\lvert z_3 - z_1\rvert$ correctly	2
• evaluates $\lvert z_2 - z_1\rvert$ or $\lvert z_3 - z_1\rvert$ correctly or significant progress towards both	1

Sample Answer:

From (iii), $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$. Whereas $\cos\frac{4\pi}{5} < 0$ and hence $\cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$.

$$\begin{aligned}|z_3 - z_1| &= \left|\text{cis}\tfrac{4\pi}{5} - 1\right| \\ &= \sqrt{\left(\cos\tfrac{4\pi}{5} - 1\right)^2 + \left(\sin\tfrac{4\pi}{5}\right)^2} \\ &= \sqrt{2 - 2\cos\tfrac{4\pi}{5}} = \sqrt{2 + \tfrac{1}{2} + \tfrac{\sqrt{5}}{2}} \\ &= \tfrac{\sqrt{5+\sqrt{5}}}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{Also } |z_2 - z_1| &= \left|\text{cis}\tfrac{2\pi}{5} - 1\right| \\ &= \sqrt{2 - 2\cos\tfrac{2\pi}{5}} \\ &= \sqrt{2 + \tfrac{1}{2} - \tfrac{\sqrt{5}}{2}} \\ &= \tfrac{\sqrt{5-\sqrt{5}}}{\sqrt{2}}\end{aligned}$$

$$\text{Hence } \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \frac{\sqrt{5+\sqrt{5}}}{\sqrt{5-\sqrt{5}}} = \sqrt{\frac{\sqrt{5}+1}{\sqrt{5}-1}} = \sqrt{\frac{(\sqrt{5}+1)^2}{4}} = \frac{\sqrt{5}+1}{2}\text{, as required.}$$

Question 14(b) (i) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Mark
• correct solution	1

Sample Answer:
When $t = 3$, $\underset{\sim}{r} = -\frac{1}{\sqrt{2}}\underset{\sim}{i} + 0\underset{\sim}{j} + \frac{1}{\sqrt{2}}\underset{\sim}{k}$

Question 14(b) (ii) (3 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution, NB there are many correct vector equations for the line	3
• evaluates direction vector $\underset{\sim}{v}(3)$ or a scalar multiple of the direction vector	2
• differentiates correctly to find $\underset{\sim}{v}(t)$	1

Sample Answer:
$$\underset{\sim}{v}(t) = \frac{\pi}{4}\left[-\sin\left(\frac{\pi}{4}t\right)\underset{\sim}{i} + \left(-\sin\left(\frac{\pi}{4}t\right) + \cos\left(\frac{\pi}{4}t\right)\right)\underset{\sim}{j} + \cos\left(\frac{\pi}{4}t\right)\underset{\sim}{k}\right]$$
$$\underset{\sim}{v}(3) = \frac{\pi}{4}\left[-\frac{1}{\sqrt{2}}\underset{\sim}{i} + \left(-\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}\right)\underset{\sim}{j} + \frac{1}{\sqrt{2}}\underset{\sim}{k}\right] = \frac{\pi}{4\sqrt{2}}\begin{pmatrix}-1\\-2\\1\end{pmatrix}$$

Therefore the tangent will be given by: $\underset{\sim}{r} = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$, for $\lambda \in \mathbb{R}$.

Question 14(c) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	3
• correctly completes two substitutions or one substitution with correct limits	2
• correctly completes at least one relevant substitution with or without limits	1

Sample Answer:
Attempting with a single substitution:

Let $u^2 = 1 + \sqrt{x}$
$$2u\,du = \frac{1}{2\sqrt{x}}dx$$
$$4u\sqrt{x}\,du = dx$$
$$4u\left(u^2 - 1\right)du = dx$$

$$\int_0^9 \frac{1}{\sqrt{1+\sqrt{x}}}\,dx = \int_1^2 \frac{1}{u} \times 4u\left(u^2-1\right)du$$
$$= 4\int_1^2 \left(u^2-1\right)du$$
$$= 4\left[\frac{u^3}{3} - u\right]_1^2$$
$$= 4\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right] = \frac{16}{3}$$

Question 15 (15 marks)

Question 15(a) (4 marks)

Outcomes Assessed:* *MEX12-5

Targeted Performance Bands:* *E3-E4

Criteria	Marks
• correct solution	4
• two correct applications of integration by parts to arrive at $\int f(x)\,dx$	3
• one correct application of integration by parts	2
• correct x-intercepts	1

Sample Answer:

$f(x) = e^{-x}\cos x, \quad x \in \left[0, \frac{3\pi}{2}\right]$. Now, the x-intercepts at $e^{-x}\cos x = 0$, hence at $x = \frac{\pi}{2}$ then $\frac{3\pi}{2}$, and since $e^{-x} > 0$, the sign of the cosine function will be the sign of the function, so the curve is above the x-axis for $0 < x < \frac{\pi}{2}$ and below for for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

So we're looking to find the ratio of integrals $\int_0^{\frac{\pi}{2}} f(x)\,dx : -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\,dx$.

$$\int e^{-x}\cos x\,dx \qquad dv = e^{-x}\,dx \quad \text{and} \quad u = \cos x$$

$$v = -e^{-x} \qquad du = -\sin x\,dx$$

Hence, $\int e^{-x}\cos x\,dx = \left[-e^{-x}\cos x\right] - \int -e^{-x} \times -\sin x\,dx$

$$= -e^{-x}\cos x - \int e^{-x} \times \sin x\,dx$$

$$dv = e^{-x}\,dx \quad \text{and} \quad u = \sin x$$

$$v = -e^{-x} \qquad du = \cos x\,dx$$

Hence, $\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \left(-e^{-x}\sin x - \int -e^{-x} \times \cos x\,dx\right)$

So, $2\int e^{-x}\cos x\,dx = -e^{-x}\cos x + e^{-x}\sin x + c_1$

Finally, $\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + c$

Ratio of areas will be given by

$$\int_0^{\frac{\pi}{2}} f(x)\,dx : -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\,dx = \left[\tfrac{1}{2}e^{-x}(\sin x - \cos x)\right]_0^{\frac{\pi}{2}} : -\left[\tfrac{1}{2}e^{-x}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left[e^{-\frac{\pi}{2}}(1-0) - e^{0}(0-1)\right] : -\left[e^{-\frac{3\pi}{2}}(-1-0) - e^{-\frac{\pi}{2}}(1-0)\right]$$

$$= \left(e^{-\frac{\pi}{2}} + 1\right) : \left(e^{-\frac{3\pi}{2}} + e^{-\frac{\pi}{2}}\right) \quad \text{, now multiply both terms by } e^{\frac{3\pi}{2}}$$

$$= \left(e^{\pi} + e^{\frac{3\pi}{2}}\right) : (1 + e^{\pi}) = \frac{e^{\pi}\left(e^{\frac{\pi}{2}} + 1\right)}{e^{\pi} + 1}, \text{ as required.}$$

Question 15(b) (4 marks)
***Outcomes Assessed:* MEX12-6, MEX12-7**
***Targeted Performance Bands:* E3-E4**

Criteria	Marks
• correct solution	4
• uses initial conditions to find x as a function of v	3
• integrates to find an expression for x wrt v, even without constant	2
• correct $\dot{v}$	1

Sample Answer:

Gravity acts in the direction of the slope with magnitude $10\cos 60^\circ = 5\ \text{ms}^{-2}$.
Also since $F = ma$, the acceleration due to friction is $\frac{v}{20}$.

$$\dot{v} = -5 - \tfrac{v}{20}$$
$$v\frac{dv}{dx} = -\frac{100+v}{20}$$
$$\frac{dx}{dv} = -\frac{20v}{100+v}$$
$$x = -\int \frac{20v}{100+v}\,dv$$
$$-x = 20\int\left(\frac{v+100}{v+100} - \frac{100}{v+100}\right)dv$$
$$-x = 20\left(v - 100\ln|v+100|\right) + c$$

When $x = 0$, $v = 144$ km/h $= 40$ m/s, hence

$$0 = 20(40 - 100\ln 140) + c$$
$$c = 20(100\ln 140 - 40)$$
$$-x = 20\left(v - 100\ln|v+100| + 100\ln 140 - 40\right)$$
$$x = 20\left(40 - v + 100\ln\left|\frac{v+100}{140}\right|\right)$$

When $v = 0$, $x = 20\left(40 + 100\ln\frac{5}{7}\right) \approx 127$ metres.

Question 15(c) (i) (2 marks)
***Outcomes Assessed:* MEX12-7**
***Targeted Performance Bands:* E2-E3**

Criteria	Marks
• correct solution	2
• significant progress towards correct solution	1

Sample Answer:

From the Reference Sheet: $\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$\begin{aligned}\text{So, RHS } &= 2\sin((n-1)x)\sin x\\ &= \cos((n-1)x - x) - \cos(((n-1)x + x)\\ &= \cos(((n-2)x) - \cos(nx) = \text{ LHS}\end{aligned}$$

Question 15(c) (ii) (2 marks)
Outcomes Assessed: *MEX12-7*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	2
• uses identity in (i) to find simple sine integrand or similar progress	1

Sample Answer:

$$\begin{aligned}\text{LHS} &= I_n - I_{n-2}\\ &= \int \frac{\cos nx}{\sin x}\,dx - \int \frac{\cos(n-2)x}{\sin x}\,dx\\ &= -\int \frac{\cos((n-2)x) - \cos(nx)}{\sin x}\,dx, \quad \text{and from (i):}\\ &= 2\int -\sin((n-1)x)\,dx\\ &= 2\left[\frac{1}{n-1}\times\cos((n-1)x)\right] + c\\ &= \text{RHS}\end{aligned}$$

Question 15(c) (iii) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• uses (ii) to find an expression in x for the integral	2
• expresses integral in terms of I_n	1

Sample Answer:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{\cos 2x - \cos 6x}{\sin x}\,dx &= \Big[I_2 - I_6\Big]_0^{\frac{\pi}{3}}\\ &= \Big[-(I_6 - I_4) - (I_4 - I_2)\Big]_0^{\frac{\pi}{3}}, \quad \text{and from (ii):}\\ &= -\left[\frac{2\cos(5x)}{5} + \frac{2\cos(3x)}{3}\right]_0^{\frac{\pi}{3}}\\ &= -\left[\frac{2}{5}\left(\frac{1}{2}-1\right) + \frac{2}{3}(-1-1)\right]\\ &= \frac{1}{5} + \frac{4}{3} = \frac{23}{15}\end{aligned}$$

Question 16 (15 marks)

Question 16(a) (3 marks)

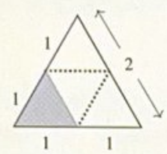
Outcomes Assessed: *MEX12-7, MEX12-8*

Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Completes the proof correctly	3
• Correctly illustrates the figure for $n = k$	2
• Demonstrates the result true for $n = 1$	1

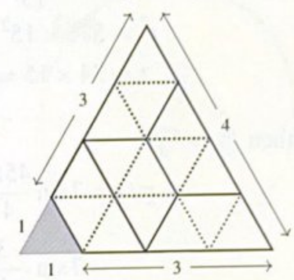
Sample Answer:

In the case of $n = 1$, the triangle of side length $2^1 = 2$ has one triangle left over, as shaded.



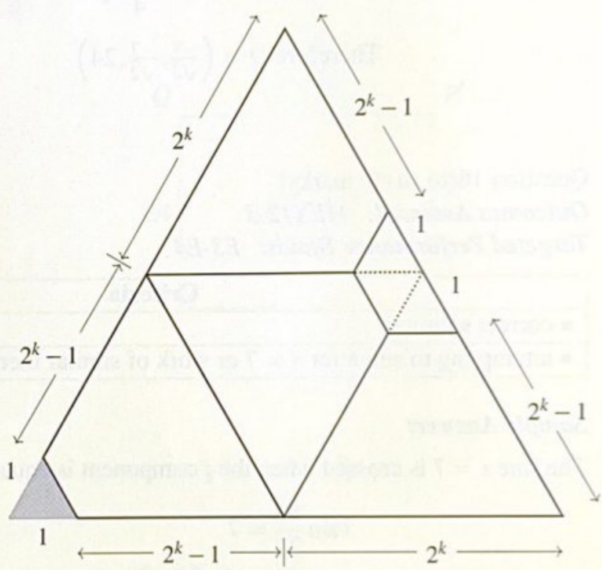
Hence the result is true for $n = 1$.

Not to contribute to the proof, but the case of $n = 2$ is pictured, with total side length of $2^2 = 4$ and 1 missing triangle, as shaded.



Assume the result is true for some integer $n = k$, which means it is possible to create a triangle of side length 2^k with one triangle of side length 1 missing in the corner.

Now take four of these triangles of side length 2^k and arrange them as in the diagram on the right. Observe that one of the tiles can fill the gap left by three of the missing corner triangles on the right.



Thus an equilateral triangle with sides $2^k + 2^k = 2^{k+1}$ with one triangle of side length 1 missing in the corner.

Hence the result is true by the Principle of Mathematical Induction.

Question 16(b) (i) (3 marks)
Outcomes Assessed: MEX12-3
Targeted Performance Bands: E3-E4

Criteria	Marks
• correct solution	3
• evaluates t for Q	2
• substitutes $\|OQ\| = 25$ into $\|\underset{\sim}{r}\|$	1

Sample Answer:

$$|OQ|^2 = 25^2 = 49\sin^2\left(\frac{\pi t}{32}\right) + 49\cos^2\left(\frac{\pi t}{32}\right) + \left(\frac{t}{15}\right)^2$$
$$625 = 49 + \frac{t^2}{15^2}$$
$$t^2 = 576 \times 15^2$$
$$t = 24 \times 15 = 360 \text{ seconds}$$

So, if $t = 360$, then $\frac{\pi t}{32} = \frac{45\pi}{4}$

$$\underset{\sim}{r}(t) = 7\sin\frac{45\pi}{4}\underset{\sim}{i} + 7\cos\frac{45\pi}{4}\underset{\sim}{j} + \frac{360}{15}\underset{\sim}{k}$$
$$= 7\sin\frac{-3\pi}{4}\underset{\sim}{i} + 7\cos\frac{-3\pi}{4}\underset{\sim}{j} + 24\underset{\sim}{k}$$

Therefore $Q = \left(\frac{-7}{\sqrt{2}}, \frac{-7}{\sqrt{2}}, 24\right)$

Question 16(b) (ii) (2 marks)
Outcomes Assessed: MEX12-3
Targeted Performance Bands: E3-E4

Criteria	Marks
• correct solution	2
• attempting to solve for $x = 7$ or work of similar merit	1

Sample Answer:

The line $x = 7$ is crossed when the $\underset{\sim}{i}$ component is equal to 7.

$$7\sin\frac{\pi t}{32} = 7$$
$$\frac{\pi t}{32} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$
$$t = 16, 80, 144, 208, 272, 336, 400, \cdots$$

Hence line $x = 7$ is crossed 6 times in the 360 seconds it takes the ant to crawl to Q.

Question 16(b) (iii) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E4*

Criteria	Marks
• correct solution	2
• significant progress towards correct solution, for example finding the arc length QP', or conceiving of the ant's path as a rectangle	1

Sample Answer:

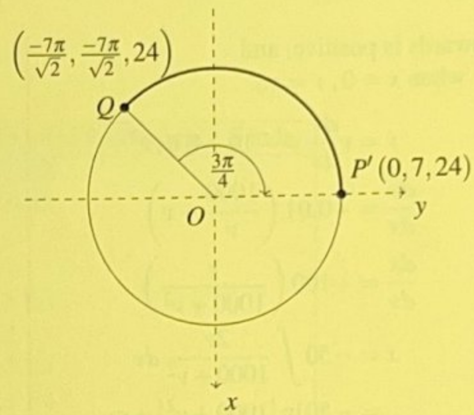
As illustrated in the diagrams on the right, the most direct path from Q to P can be found by considering the cylindrical column as a rectangle with Q and P as opposite vertices.

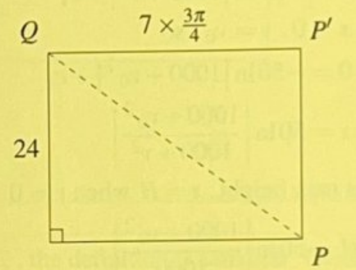
Let the point P' be at the same height of Q and directly above P. Hence $P' = (0,7,24)$, and the points Q and P' lie on the plane $z = 24$, as in the first diagram on the right.

The arc length from Q to P' will be $7 \times \frac{3\pi}{4}$. This is also represented in the second diagram, and this gives:

$$\text{The ant's path QP} = \sqrt{24^2 + \left(\frac{21\pi}{4}\right)^2}$$
$$\approx 29.1\,\text{cm}$$

ABOVE VIEW





Question 16(c) (5 marks)
Outcomes Assessed: *MEX12-6, MEX12-7*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	5
• forming an equation with only v_0	4
• one correct integration on the downward journey	3
• correct substitution of initial conditions	2
• one correct integration on the upward journey	1

Sample Answer:

Upwards is positive, and when $x = 0$, $v = v_0$.

$$\dot{v} = v\frac{dv}{dx} = -10 - 0.01v^2$$
$$\frac{dv}{dx} = -0.01\left(\frac{1000}{v} + v\right)$$
$$\frac{dx}{dv} = -100\left(\frac{v}{1000 + v^2}\right)$$
$$x = -50\int \frac{2v}{1000 + v^2}\,dv$$
$$= -50\ln\left|1000 + v^2\right| + c_1$$

when $x = 0$, $v = v_0$, so

$$0 = -50\ln\left|1000 + {v_0}^2\right| + c_1$$
$$x = 50\ln\left|\frac{1000 + {v_0}^2}{1000 + v^2}\right|$$

Reaches max height, $x = H$ when $v = 0$

$$H = 50\ln\left|\frac{1000 + {v_0}^2}{1000}\right|$$

On the downward journey let downwards be positive, and when $x = 0$, $v = 0$.

$$\dot{v} = v\frac{dv}{dx} = 10 - 0.01v^2$$
$$\frac{dv}{dx} = 0.01\left(\frac{1000}{v} - v\right)$$
$$\frac{dx}{dv} = 100\left(\frac{v}{1000 - v^2}\right)$$
$$x = -50\int \frac{-2v}{1000 - v^2}\,dv$$
$$= -50\ln\left|1000 - v^2\right| + c_2$$

when $x = 0$, $v = 0$, so

$$0 = -50\ln|1000| + c_2$$
$$x = 50\ln\left|\frac{1000}{1000 - v^2}\right|$$

On impact, $x = H$, and crucially $v = \frac{v_0}{7}$

$$50\ln\left|\frac{1000 + {v_0}^2}{1000}\right| = 50\ln\left|\frac{1000}{1000 - (v_0/7)^2}\right|$$

$$(1000 + {v_0}^2)\left(1000 - \left(\tfrac{v_0}{7}\right)^2\right) = 1000^2$$
$$1000{v_0}^2 - \frac{1000}{49}{v_0}^2 - \frac{1}{49}{v_0}^4 = 0$$
$${v_0}^2\left({v_0}^2 + 1000 - 49000\right) = 0$$

Hence $v_0 = 0$ or $\sqrt{48000} \approx 219.1\,\mathrm{ms}^{-1}$.