CSSA

2022
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 1

Afternoon Session
Friday, 12 August 2022

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black pen
- NESA-approved calculators may be used
- A reference sheet is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 70

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks

Attempt Questions 1–10
Allow about 15 minutes for this section
Use the Multiple-Choice Answer Sheet for Questions 1–10

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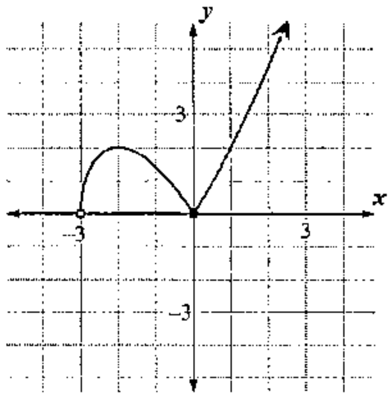
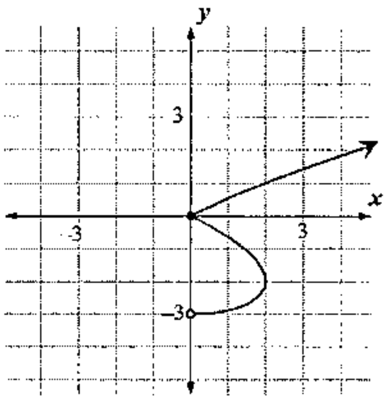
1 Below is the graph of a function.

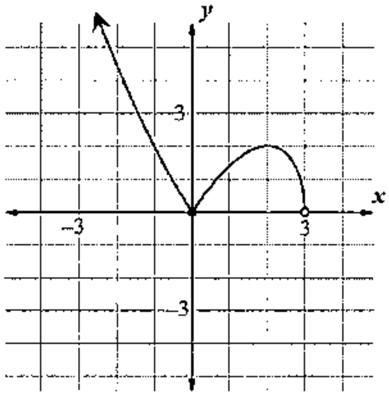


Which of the following is the correct graph of its inverse?

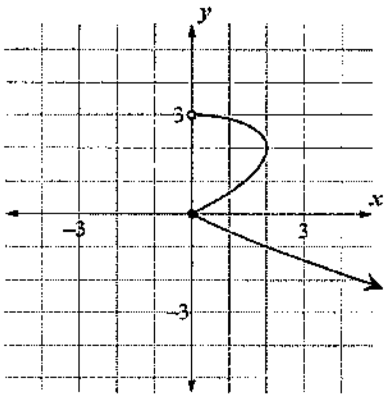
A.



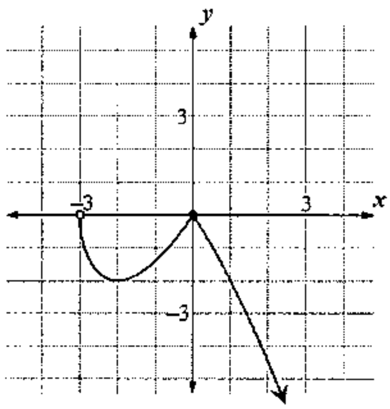
B.



C.



D.



2 Which of the following expressions is equivalent to $\int \frac{-2}{\sqrt{1-4x^2}}\,dx$?

A. $\frac{1}{2}\sin^{-1}(2x)+c$

B. $\sin^{-1}(2x)+c$

C. $\frac{1}{2}\cos^{-1}(2x)+c$

D. $\cos^{-1}(2x)+c$

3 Marie has attempted to differentiate the inverse cosine function, but has made a mistake in her working below.

$$y=\cos^{-1}x$$

$$x=\cos y \qquad \textbf{line A}$$

$$\frac{dx}{dy}=\sin y \qquad \textbf{line B}$$

$$\frac{dy}{dx}=\frac{1}{\sin y} \qquad \textbf{line C}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\cos^2 y}} \qquad \textbf{line D}$$

$$=\frac{1}{\sqrt{1-x^2}}$$

In which line has she made her mistake?

A. line A

B. line B

C. line C

D. line D

4 Which of the following expressions is NOT equivalent to $2\sin\alpha\sin 2\alpha$?

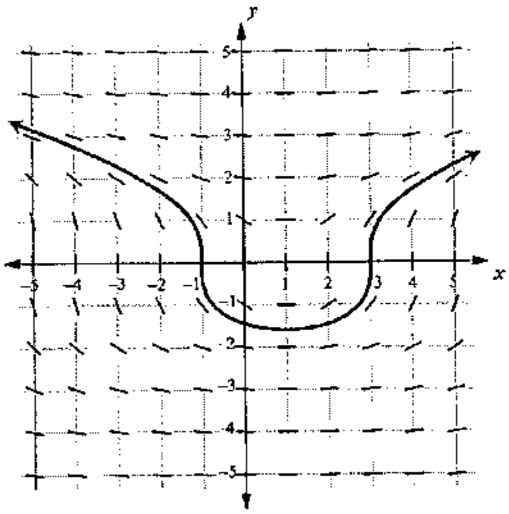
A. $4\left(\cos\alpha-\cos^3\alpha\right)$

B. $4\cos\alpha\sin^2\alpha$

C. $\cos\alpha-\cos 3\alpha$

D. $2(\cos\alpha-\cos 3\alpha)$

5 The direction field diagram below has a particular solution shown by the curve sketched on it.



Which of the following differential equations and Cartesian equations satisfies the solution curve sketched on this direction field?

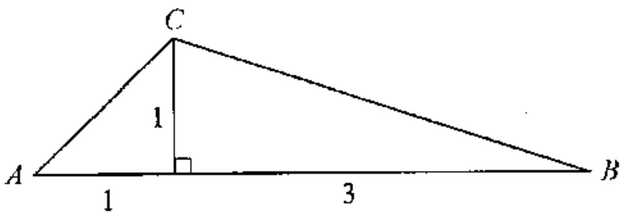
A. $\dfrac{dy}{dx} = \dfrac{2x-2}{3y^2}$ and $y^3 = x^2 - 2x + 3$

B. $\dfrac{dy}{dx} = \dfrac{2x-2}{3y^2}$ and $y^3 = x^2 - 2x - 3$

C. $\dfrac{dy}{dx} = \dfrac{2x-2}{y^2}$ and $y^3 = 3\left(x^2 - 2x + 3\right)$

D. $\dfrac{dy}{dx} = \dfrac{2x-2}{y^2}$ and $y^3 = 3\left(x^2 - 2x - 3\right)$

6 In triangle ABC, what is the exact value of $\sin(A+B)$?



A. $\dfrac{2}{\sqrt{5}}$

B. $\dfrac{\sqrt{5}}{2}$

C. $\dfrac{5}{\sqrt{5}}$

D. $\dfrac{\sqrt{5}}{5}$

7 Which of the following is an odd function?

A. $a(x) = \frac{\pi}{2} + \cos^{-1} x$

B. $b(x) = \frac{\pi}{2} - \cos^{-1} x$

C. $c(x) = \frac{\pi}{2} + \cos^{-1}(1-x)$

D. $d(x) = \frac{\pi}{2} - \cos^{-1}(1-x)$

8 Which of the following is the length of the projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$?

A. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{\underset{\sim}{a} \cdot \underset{\sim}{a}}$

B. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{\underset{\sim}{b} \cdot \underset{\sim}{b}}$

C. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{a}|}$

D. $\dfrac{|\underset{\sim}{a} \cdot \underset{\sim}{b}|}{|\underset{\sim}{b}|}$

9 Consider the vectors $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.
If $\underset{\sim}{c} = \dfrac{2\underset{\sim}{b} + \underset{\sim}{a}}{3}$, which of the following is true?

A. $AC = 2BC$

B. $2AC = BC$

C. $2AC = 3BC$

D. $3AC = 2BC$

10 Let X be a random variable with a binomial distribution, with n independent trials each with probability p of success. If p changes from 0·6 to 0·5, which of the following is true?

A. The mean of X increases and the variance of X increases.

B. The mean of X increases and the variance of X decreases.

C. The mean of X decreases and the variance of X increases.

D. The mean of X decreases and the variance of X decreases.

Section II
60 marks

Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Your responses for Questions 11-14 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{3}{x-2} \leq 1$. **3**

(b) If $\theta = \sin^{-1}\dfrac{1}{3}$, show that $\cos 2\theta = \dfrac{7}{9}$. **2**

(c) Using the substitution $u = \sqrt{x}$ evaluate $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$. **3**

(d) Given $\underset{\sim}{a}$ is a unit vector and $(\underset{\sim}{x} - \underset{\sim}{a}) \cdot (\underset{\sim}{x} + \underset{\sim}{a}) = 24$, evaluate $|\underset{\sim}{x}|$. **2**

(e) (i) Write the expression $2\sqrt{3}\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 \leq \alpha \leq \dfrac{\pi}{2}$. **2**

(ii) Hence find the value of k, where $0 \leq k \leq 2\pi$, for which **3**

$$\int_0^k \left(2\sqrt{3}\cos x - 2\sin x\right) dx = 2.$$

Question 12 (14 marks) Use a SEPARATE writing booklet.

(a) Emma is the driver of a car and is planning to take six of her friends to a local soccer match. She can take at most four passengers in her car. She will need to drive two trips. **2**

Determine the number of different ways in which her friends can be selected for the first trip.

(b) In each round of a game, the player rolls five standard six-sided dice. They win the round if exactly two dice show a result of 2. Out of 1000 rounds, what is the expected number of wins? **2**

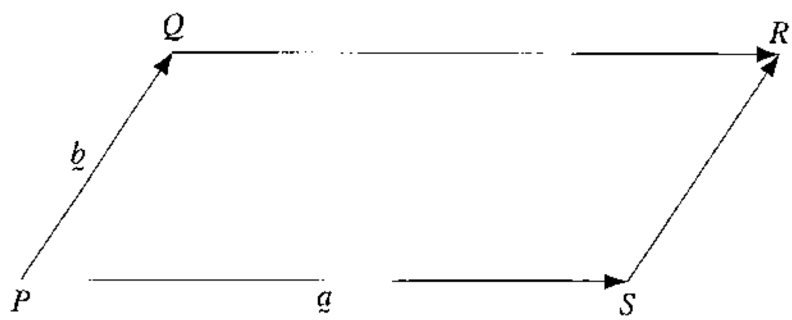
(c) Given the pair of parametric equations $x = 3\cos^2\theta$ and $y = \sin\theta\cos\theta$ for $\theta \in \left[0, \frac{\pi}{2}\right]$, find a Cartesian equation in the form $y = f(x)$. **3**

(d) Prove by mathematical induction that **3**

$$1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = 2 + (n-1)2^{n+1}$$

for all positive integers n.

(e) For the quadrilateral $PQRS$, $\overrightarrow{PS} = \overrightarrow{QR} = \underset{\sim}{a}$ and $\overrightarrow{PQ} = \overrightarrow{SR} = \underset{\sim}{b}$, as shown in the diagram below. **4**



If $\left|\overrightarrow{PR}\right| = \left|\overrightarrow{SQ}\right|$, use vector methods to prove that $PQRS$ is a rectangle.

Question 13 (16 marks) Use a SEPARATE writing booklet.

(a) The first camel arrived in Australia in 1840. Camels were released into the wild in the 1920s and have become a pest that is a threat to fragile native ecosystems. The growth rate of the population P millions of wild camels in Central Australia at time t months after the end of 2021 is given by

$$\frac{dP}{dt} = \frac{1}{80}P\left(1 - \frac{P}{3}\right).$$

The wild camel population was reported to be 1 million at the end of 2021. That is, when $t = 0$, $P = 1$.

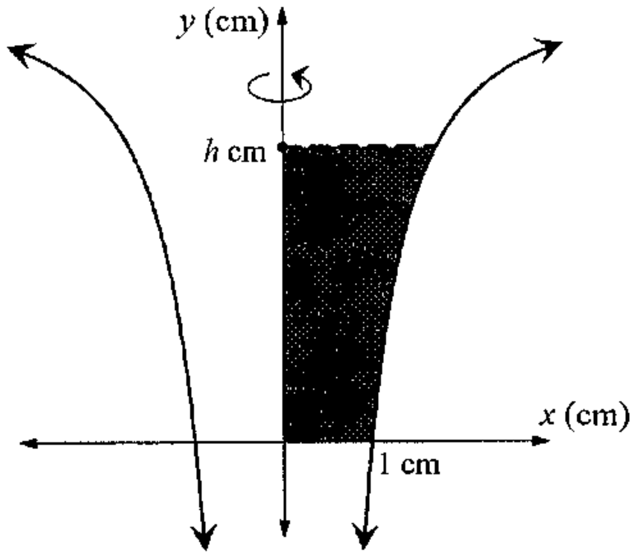
(i) Show that $\dfrac{3}{P(3-P)} = \dfrac{1}{P} + \dfrac{1}{3-P}$. **1**

(ii) Solve the differential equation to show that $P = \dfrac{3}{2e^{-\frac{t}{80}} + 1}$. **3**

(iii) Write down the limiting population of the wild camels. **1**

(iv) How many years will it take for the population to become double what it was at the end of 2021? **2**

(b) Consider the shaded region bounded by the curve $y = 9 - \dfrac{9}{x^2}$, the coordinate axes, and the line $y = h$ as shown in the diagram below. **4**



A hollow vase is generated by rotating the section of this curve in the first quadrant about the y-axis. The vase is being filled with water at a constant rate of $50\,\text{cm}^3/\text{min}$. Find the depth h cm of the water at the instant when it is rising at a rate of $\dfrac{50}{3\pi}$ cm/min.

Question 13 continues on page 9

Question 13 (continued)

(c) To offset the global rise in carbon emissions, Carolyn decided to re-grow a native forest on a large property in western NSW previously used for grazing sheep. She removed the sheep and waited for the forest to regenerate naturally.

It has now been 10 years, and 30% of the property is now forested. Carolyn needs to return to the property to survey the types and sizes of the tree species present in the forested areas. She will use this data to calculate the amount of carbon that has been captured on the property in the last 10 years.

To do the survey, Carolyn needs to generate a random sample of points on the property. Let $\hat{P}$ be the proportion of sampled points which are forested.

(i) If she generates a random sample of 100 points on the property, what is the probability that $\hat{P}$ exceeds 25%? **2**

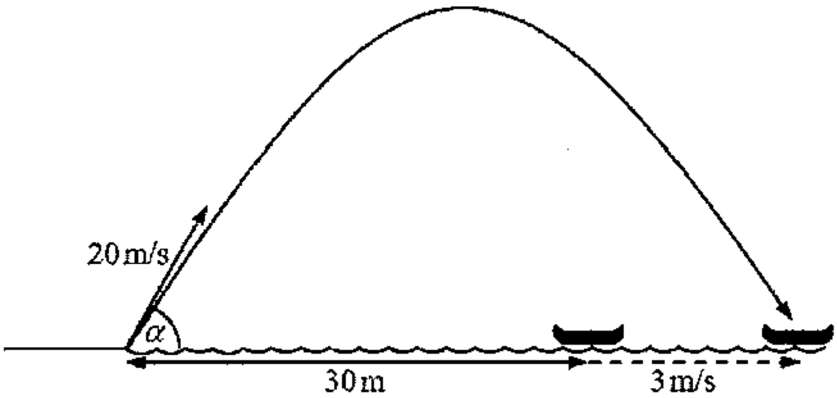
(ii) Using Table 1 below, or otherwise, find the smallest number of points she needs to sample such that $P\left(\hat{P} > 0{\cdot}25\right)$ is at least 95·5%. **3**

Table 1: Single decimal place table of $P(Z \le z)$

z	+·0	+·1	+·2	+·3	+·4	+·5	+·6	+·7	+·8	+·9
0	0·500	0·540	0·579	0·618	0·655	0·691	0·726	0·758	0·788	0·816
1	0·841	0·864	0·885	0·903	0·919	0·933	0·945	0·955	0·964	0·971
2	0·977	0·982	0·986	0·989	0·992	0·994	0·995	0·997	0·997	0·998

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A projectile is thrown at 20 m/s at an angle of inclination α from a lake's edge towards a canoe that is 30 metres away and moving away from shore at 3 m/s as shown in the diagram below.



The position vector $\underset{\sim}{r}(t)$ of the projectile t seconds after it is thrown is given by

$$\underset{\sim}{r}(t) = \begin{bmatrix} 20t\cos\alpha \\ -5t^2 + 20t\sin\alpha \end{bmatrix} \qquad \text{Do NOT prove this.}$$

(i) Show that the projectile lands in the canoe if $20\sin 2\alpha - 6\sin\alpha - 15 = 0$. **2**

(ii) Find integers a, b, and c such that **1**

$$15x^4 + 92x^3 + 30x^2 - 68x + 15 = (3x^2 + 16x - 5)(ax^2 + bx + c).$$

(iii) Using the factorisation in (ii) and the substitution $x = \tan\left(\frac{\alpha}{2}\right)$, find solutions for α so that the projectile lands in the canoe. Give your answers correct to the nearest degree. **3**

Question 14 continues on page 11

Question 14 (continued)

(b) Consider the polynomial $P(x) = x^3 + kx^2 + mx + n$ with distinct zeros α, β and γ.

Let $Q(x)$ be a monic polynomial with zeros α^2, β^2 and γ^2.

(i) Show that $Q(x^2) = -P(x)P(-x)$. **2**

(ii) Hence show that $Q(x) = x^3 - (k^2 - 2m)x^2 + (m^2 - 2kn)x - n^2$. **2**

(c) Consider the polynomial $P(x) = x^3 - 3x^2 - x + 1$ with distinct zeros α, β and γ.

(i) By using two applications of the result in part (b) (ii), or otherwise, show that the polynomial $R(x) = x^3 - 107x^2 + 27x - 1$ has zeros α^4, β^4 and γ^4. **2**

(ii) By assuming that $\alpha > 0$, and that the magnitude of α is significantly greater than the magnitudes of β and γ, and by considering the sum of zeros of $R(x)$ above, explain why $\alpha^4 \approx 107$. **1**

(iii) Hence find an approximation for α to 2 decimal places and verify it is a good approximation for a zero of $P(x)$. **2**

End of Examination

EXAMINERS

David Houghton (Convenor)	Oxley College, Burradoo
Geoff Carroll	Sydney Grammar School, Darlinghurst
Rebekah Johnson	Loreto Kirribilli, Kirribilli
Svetlana Onisczenko	Meriden School, Strathfield
Gerry Sozio	Edmund Rice College, West Wollongong

2022

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

CSSA

MARKING GUIDELINES

Mathematics Extension I

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	A	ME11-1, ME11-2	E2
2	D	ME12-4	E2
3	B	ME12-7	E2-E3
4	D	ME11-3	E2-E3
5	B	ME12-4	E2-E3
6	A	ME11-3	E2-E3
7	B	ME11-1	E3
8	C	ME12-2	E3-E4
9	A	ME12-2	E3-E4
10	C	ME12-5	E4

Question 1 (1 mark)
Outcomes Assessed: *ME11-1, ME11-2*
Targeted Performance Bands: *E2*

Solution	Mark
Reflect curve over the line $y = x$ Hence A	1

Question 2 (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E2*

Solution	Mark
Using $\frac{d}{dx}\cos^{-1} f(x)$ on reference sheet: $\int \frac{-2}{\sqrt{1-4x^2}} = \cos^{-1}(2x) + c$. Hence D	1

Question 3 (1 mark)
***Outcomes Assessed:** ME12-7*
***Targeted Performance Bands:** E2-E3*

Solution	Mark
In Line B, $\frac{d}{dy}(\cos y) = -\sin y \neq \sin y$. Hence B	1

Question 4 (1 mark)
***Outcomes Assessed:** ME11-3*
***Targeted Performance Bands:** E2-E3*

Solution	Mark
Analysing the options, A=B while C≠D, so the answer must be C or D. $2\sin\alpha\sin 2\alpha = 2\times\frac{1}{2}[\cos(\alpha-2\alpha)-\cos(\alpha+2\alpha)]$ $= \cos(-\alpha)-\cos(3\alpha)$ $= \cos\alpha - \cos 3\alpha$ (C) $\neq 2(\cos\alpha - \cos 3\alpha)$ (D) But also, $2\sin\alpha\sin 2\alpha = 2\sin\alpha\times(2\sin\alpha\cos\alpha)$ $= 4\sin^2\alpha\cos\alpha$ (B) $= 4\cos\alpha\left(1-\cos^2\alpha\right)$ $= 4\left(\cos\alpha - \cos^3\alpha\right)$ (A) Hence D	1

Question 5 (1 mark)
***Outcomes Assessed:** ME12-4*
***Targeted Performance Bands:** E2-E3*

Solution	Mark
Checking the y-intercepts: A gives $y = \sqrt[3]{3} > 0$, which does NOT match the graph. B gives $y = \sqrt[3]{-3} \approx -1{\cdot}4$, which does match the graph. C gives $y = \sqrt[3]{9} > 0$, which does NOT match the graph. B gives $y = \sqrt[3]{-9} < -2$, which does NOT match the graph. Hence B	1

Question 6 (1 mark)
Outcomes Assessed: *ME11-3*
Targeted Performance Bands: *E2-E3*

Solution	Mark
$\sin(A+B)=\sin A\cos B+\cos A\sin B=\frac{1}{\sqrt{2}}\times\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{10}}$ $=\frac{4}{\sqrt{20}}=\frac{2}{\sqrt{5}}$ Hence A	1

Question 7 (1 mark)
Outcomes Assessed: *ME11-1*
Targeted Performance Bands: *E3*

Solution	Mark
Note the complementary angle result $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$. $a(x)=\frac{\pi}{2}+\left(\frac{\pi}{2}-\sin^{-1}x\right)=\pi-\sin^{-1}x$ which is NOT an odd function. So, $b(x)=\frac{\pi}{2}-\left(\frac{\pi}{2}-\sin^{-1}x\right)=\sin^{-1}x$ which is an odd function. The domain of $\cos^{-1}x$ is $-1\le x\le 1$ so the domain of $c(x)$ and $d(x)$ will be given by: $-1\le 1-x\le 1$ $-2\le -x\le 0,$ $0\le x\le 2.$ which makes it impossible for these to be odd functions. Hence B	1

Question 8 (1 mark)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E3-E4*

Solution	Mark
$\text{proj}_{a}b=\frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert^2}\underset{\sim}{a}=\frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}\frac{\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}$ Where $\frac{\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}$ is a unit vector in the same direction as $\text{proj}_{a}b$. So the length of projection equal the length of $\frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}=\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert}$, but since the dot product can be negative, the length will be $\frac{\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{a}\rvert}$. Hence C	1

Question 9 (1 mark)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E3-E4*

Solution	Mark
$\overrightarrow{AC} = \dfrac{2\underset{\sim}{b}+\underset{\sim}{a}}{3} - \underset{\sim}{a} = \dfrac{2(\underset{\sim}{b}-\underset{\sim}{a})}{3}$ $\overrightarrow{BC} = \dfrac{2\underset{\sim}{b}+\underset{\sim}{a}}{3} - \underset{\sim}{b} = \dfrac{\underset{\sim}{a}-\underset{\sim}{b}}{3}$ Hence $\lvert\overrightarrow{AC}\rvert = 2\lvert\overrightarrow{BC}\rvert$ That is, $AC = 2BC$ Hence A	1

Question 10 (1 mark)
Outcomes Assessed: *ME12-5*
Targeted Performance Bands: *E4*

Solution	Mark
Since the mean of X is np, if p decreases, the mean decreases. The variance of X is $\sigma^2 = npq$. When $p = 0{\cdot}6$, $\sigma^2 = 0{\cdot}6 \times 0{\cdot}4n = 0{\cdot}24n$ When $p = 0{\cdot}6$, $\sigma^2 = 0{\cdot}5 \times 0{\cdot}5n = 0{\cdot}25n$ So the variance increases when p changes from 0·6 to 0·5 . Hence C	1

Section II

60 marks

Question 11 (15 marks)

11(a) (3 marks)
Outcomes Assessed: *ME11-2, ME11-7*
Targeted Performance Bands: *E2*

Criteria	Marks
• Provides correct solution, either algebraically or graphically	3
• Multiplies both sides by $(x-2)^2$ to create a quadratic inequality OR correctly graphs the functions $y = \frac{3}{x-2}$ and $y = 1$ OR other valid method	2
• Makes some progress towards an algebraic or graphical solution	1

Sample Answer:

$\dfrac{3}{x-2} \le 1$ Note $x \neq 2$

Multiplying both sides by $(x-2)^2$
gives $3(x-2) \le (x-2)^2$

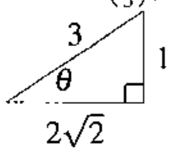
$0 \le (x-2)^2 - 3(x-2)$
$0 \le (x-2)\,[(x-2)-3]$
$0 \le (x-2)(x-5)$
Which has solution $x < 2$ or $x \ge 5$.

11(b) (2 marks)
Outcomes Assessed: *ME11-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• Provides correct solution	2
• Finds a correct right-angled triangle and expression for $\sin\theta$ and $\cos\theta$ OR uses $\cos 2\theta = 1 - 2\sin^2\theta$	1

Sample Answer:

If $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, we have the triangle:



So $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2$$

$$= \frac{8}{9} - \frac{1}{9} = \frac{7}{9} \quad \text{as required.}$$

11(c) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	3
• Correct substitution but does not calculate new limits of integration correctly	2
• Makes some progress towards the correct substitution, even with errors	1

Sample Answer:

If $u = \sqrt{x}$ then $x = u^2$ and $dx = 2u\,du$.
When $x = 0$, $u = 0$ and when $x = \pi^2$, $u = \pi$.

$$\text{So } \int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = \int_0^{\pi} \frac{\sin u}{u} \times 2u\,du$$

$$= \int_0^{\pi} 2\sin u\,du$$

$$= -2\Big[\cos u\Big]_0^{\pi}$$

$$= -2[\cos\pi - \cos 0]$$

$$= -2[-1-1]$$

$$= 4$$

11(d) (2 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E2*

Criteria	Marks
• Provides correct solution	2
• Correct evaluation of dot product	1

Sample Answer:

$$(\underset{\sim}{x} - \underset{\sim}{a}) \cdot (\underset{\sim}{x} + \underset{\sim}{a}) = 24$$

$$\underset{\sim}{x} \cdot \underset{\sim}{x} - \underset{\sim}{a} \cdot \underset{\sim}{x} + \underset{\sim}{x} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{a} = 24$$

$$|\underset{\sim}{x}|^2 - |\underset{\sim}{a}|^2 = 24$$

$$|\underset{\sim}{x}|^2 = 24 + 1^2 \quad (\text{since } |\underset{\sim}{a}| = 1)$$

$$|\underset{\sim}{x}|^2 = 25$$

So $|\underset{\sim}{x}| = 5$ since $|\underset{\sim}{x}| \geq 0$.

11(e) (i) (2 marks)
Outcomes Assessed: *ME11-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	2
• Correct approach to equate coefficients but with a mistake	1

Sample Answer:

Let $2\sqrt{3}\sin x + 2\cos x = R\sin(x+\alpha)$, where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$

$$\begin{aligned}\text{RHS} &= R\sin(x+\alpha)\\ &= R\sin x\cos\alpha + R\sin\alpha\cos x\end{aligned}$$

Equating coefficients gives $R\cos\alpha = 2\sqrt{3}$ and $R\sin\alpha = 2$

$$\begin{aligned}R^2\cos^2\alpha - R^2\sin^2\alpha &= 12+4\\ R^2 &= 16\\ R &= \pm 4 \text{ and since } R > 0,\\ R &= 4\end{aligned}$$

$$\begin{aligned}\frac{R\sin\alpha}{R\cos\alpha} &= \frac{2}{2\sqrt{3}}\\ \tan\alpha &= \frac{1}{\sqrt{3}}\\ \alpha &= \frac{\pi}{6}\end{aligned}$$

So $2\sqrt{3}\sin x + 2\cos x = 4\sin\left(x+\dfrac{\pi}{6}\right)$.

11(e) (ii) (3 marks)
Outcomes Assessed: *ME11-3, ME12-3, ME12-4*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	3
• Evaluates integral correctly and equates with part (i) but with an error in calculation	2
• Evaluates integral correctly or error carried forward	1

Sample Answer:

$$\begin{aligned}\int_0^k \left(2\sqrt{3}\cos x - 2\sin x\right) dx &= 2\\ \Big[2\sqrt{3}\sin x + 2\cos x\Big]_0^k &= 2\\ \left(2\sqrt{3}\sin k + 2\cos k\right) - \left(2\sqrt{3}\sin 0 + 2\cos 0\right) &= 2\\ 2\sqrt{3}\sin k + 2\cos k &= 4\end{aligned}$$

from part (i):

$$\begin{aligned}4\sin\left(k+\tfrac{\pi}{6}\right) &= 4\\ \sin\left(k+\tfrac{\pi}{6}\right) &= 1\\ k+\frac{\pi}{6} &= \cdots, -\tfrac{3\pi}{2}, \tfrac{\pi}{2}, \tfrac{5\pi}{2}, \cdots\\ k &= \cdots, -\tfrac{5\pi}{3}, \tfrac{\pi}{3}, \tfrac{7\pi}{3}, \cdots\end{aligned}$$

Since $0 \le k \le 2\pi$, the only solution is $k = \dfrac{\pi}{3}$.

Question 12 (14 marks)

12(a) (2 marks)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• Provides correct solution	2
• Gives ${}^6C_4$ only	1

Sample Answer:

Emma must take 2, 3 or 4 friends in her first trip, in order to take at least 2 friends in the second trip. Therefore, the number of different ways she can choose is ${}^6C_4 + {}^6C_3 + {}^6C_2 = 50$.

12(b) (2 marks)
Outcomes Assessed: *ME12-5*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	2
• Finding the term in p^2	1

Sample Answer:

Consider the binomial expansion $(p+q)^5 = \sum_{k=1}^{5} \binom{5}{k} p^{5-k} q^k$ where $p = \frac{1}{6}$ and $q = \frac{5}{6}$.

We want exactly two sixes i.e. when $k = 3$. Accordingly, the term in p^2 is $\binom{5}{3} p^2 q^3$.

So expected wins$= 1000 \times \binom{5}{3} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^3 \approx 160{\cdot}75 \cdots \approx 161$ wins

12(c) (3 marks)
Outcomes Assessed: *ME11-1, ME11-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• Chooses positive square root because of domain of parameter θ	3
• Expresses y^2 as a function of x	2
• Writes y^2 as a function of $\cos\theta$	1

Sample Answer:

Starting with $y = \sin\theta\cos\theta$ and squaring both sides we have

$$\begin{aligned} 9y^2 &= 9\sin^2\theta\cos^2\theta \\ &= 9\left(1-\cos^2\theta\right)\cos^2\theta \\ &= 3\left(3\cos^2\theta\right) - \left(3\cos^2\theta\right)^2 \\ &= 3x - x^2 \end{aligned}$$

Solving for y gives:

$$y^2 = \frac{3x - x^2}{9}$$

$$y = \frac{\sqrt{3x - x^2}}{3}$$

noting that $y > 0$ since $\theta \in \left[0, \frac{\pi}{2}\right]$

12(d) (3 marks)
Outcomes Assessed: *ME12-1, ME12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• Completes the proof correctly	3
• Substitutes the correct expression for $n = k$	2
• Demonstrates the result true for $n = 1$	1

Sample Answer:

Step 1 Prove the result true for $n = 1$.

$$\text{LHS} = 1 \times 2 \qquad\qquad \text{RHS} = 2 + (1 - 1 \times 2^2)$$
$$= 2 \qquad\qquad\qquad = 2$$

Since LHS = RHS, the result is true for $n = 1$.

Step 2 Assume the result is true for $n = k$
That is, $1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + k \times 2^k = 2 + (k-1) \times 2^{k+1}$

Step 3 Prove the result true for $n = k + 1$
i.e. Prove that $1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + k \times 2^k + (k+1) \times 2^{k+1} = 2 + (k) \times 2^{k+2}$

$$\begin{aligned}
\text{LHS} &= 1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + k \times 2^k + (k+1) \times 2^{k+1} \\
&= 2 + (k-1) \times 2^{k+1} + (k+1) \times 2^{k+1} \qquad \text{By Step 2} \\
&= 2 + 2^{k+1}\,[k - 1 + k + 1] \\
&= 2 + 2k \times 2^{k+1} \\
&= 2 + k \times 2^{k+2} \\
&= \text{RHS as required}
\end{aligned}$$

Hence the result is true by the Principle of Mathematical Induction.

12(c) (4 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• All proof steps correctly completed with a valid conclusion	4
• Uses the dot product to show that $\overrightarrow{PS} \perp \overrightarrow{PQ}$	3
• Derives a correct expression for $\overrightarrow{PR}$ and $\overrightarrow{SQ}$	2
• Establishes that *PQRS* is a parallelogram	1

Sample Answer:

We know that $\overrightarrow{PS} = \overrightarrow{QR}$ and $\overrightarrow{PQ} = \overrightarrow{SR}$ (given).

Hence *PQRS* is a parallelogram (two pairs of opposite sides equal and parallel).

$$\overrightarrow{SQ} = \overrightarrow{PQ} - \overrightarrow{PS}$$
$$= \underset{\sim}{b} - \underset{\sim}{a}$$
$$\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR}$$
$$= \underset{\sim}{a} + \underset{\sim}{b}$$

We know that $\left|\overrightarrow{PR}\right| = \left|\overrightarrow{SQ}\right|$ (given).

Hence $\left|\overrightarrow{PR}\right|^2 = \left|\overrightarrow{SQ}\right|^2$.

$$\overrightarrow{PR} \cdot \overrightarrow{PR} = \overrightarrow{SQ} \cdot \overrightarrow{SQ}$$
$$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$$
$$\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a}$$
$$2\underset{\sim}{a} \cdot \underset{\sim}{b} = -2\underset{\sim}{a} \cdot \underset{\sim}{b}$$
$$4\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$$
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$$

Since $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, $\overrightarrow{PS} \perp \overrightarrow{PQ}$.

So *PQRS* is a parallelogram with a right angle, hence *PQRS* is a rectangle, as required.

Question 13 (16 marks)

13(a) (i) (1 mark)
Outcomes Assessed: *ME11-1*
Targeted Performance Bands: *E2*

Criteria	Mark
• Provides correct solution	1

Sample Answer:

$$RHS = \frac{1}{P} + \frac{1}{3-P} = \frac{3-P}{P(3-P)} + \frac{P}{P(3-P)} = LHS$$

13(a) (ii) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3*

Criteria	Marks
• Complete solution	3
• Substitutes initial conditions correctly	2
• Reciprocates then integrates correctly	1

Sample Answer:

$$\frac{dP}{dt} = \frac{1}{80}P\left(1 - \frac{P}{3}\right)$$
$$\frac{dt}{dP} = \frac{80}{P}\left(\frac{3}{3-P}\right)$$
$$= 80\left(\frac{1}{P} + \frac{1}{3-P}\right)$$

Integrating both sides with respect to t gives
$t = 80(\ln|P| - \ln|3-P|) + C$
When $t = 0$, $P = 1$, so $C = 80\ln 2$, hence

$$t = 80\ln\left|\frac{2P}{3-P}\right|$$

Thus $$e^{\frac{t}{80}} = \frac{2P}{3-P}$$
$$2P = 3e^{\frac{t}{80}} - Pe^{\frac{t}{80}}$$

Collecting like terms gives

$$P\left(2 + e^{\frac{t}{80}}\right) = 3e^{\frac{t}{80}}$$

Thus $$P = \frac{3e^{\frac{t}{80}}}{2 + e^{\frac{t}{80}}} \times \frac{e^{-\frac{t}{80}}}{e^{-\frac{t}{80}}}$$
$$= \frac{3}{2e^{-\frac{t}{80}} + 1}$$

13(a) (iii) (1 mark)
Outcomes Assessed: *ME11-4*
Targeted Performance Bands: *E3*

Criteria	Mark
• Correct answer	1

Sample Answer:

As $t \to \infty$, $2e^{-\frac{t}{80}} \to 0$, and $P \to \frac{3}{0+1}$. Hence the limiting population is 3 million camels.

$P = 3$ $\left(\frac{1}{2}\right)$

Disclaimer

13(a) (iv) (2 marks)

Outcomes Assessed: *ME12-4*

Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution, in years (without penalising rounding)	2
• Substitutes $P = 2$ into an equation for t	1

Sample Answer:

To find t when $P = 2$

$t = 80\ln\left(\frac{2P}{3-P}\right) = 80\ln\frac{4}{1} \approx 110{\cdot}9$

So, the population will exceed 2 million in 111 months, or 9·25 years

$-\frac{1}{2}$ rounding

13(b) (4 marks)

Outcomes Assessed: *ME11-4, ME12-4, ME12-7*

Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	4
• Correctly substitutes initial conditions into $\frac{dV}{dh}$	3
• Finds correct expression for $\frac{dV}{dh}$	2
• Finds correct expression for V	1

Sample Answer:

Making x^2 the subject of the equation gives

$$\frac{9}{x^2} = 9 - y$$

$$x^2 = \frac{9}{9-y}$$

Thus

$$V = \pi\int_0^h x^2 dy$$

$$= \pi\int_0^h \frac{9}{9-y}dy$$

Differentiating both sides gives

$$\frac{dV}{dh} = \pi\frac{9}{9-h}$$

Since $\frac{dV}{dt} = 50$ and $\frac{dh}{dt} = \frac{50}{3\pi}$, we have

$$\frac{dV}{dh} = \frac{dV}{dt} \div \frac{dh}{dt}$$

$$= 50 \div \frac{50}{3\pi}$$

$$= 3\pi$$

Accordingly,

$$3\pi = \pi\frac{9}{9-h}$$

$$3(9-h) = 9$$

$$h = 6$$

The depth of the water at the instant when it is rising at the given rate will be 6 cm.

13(c) (i) (2 marks)
Outcomes Assessed: *ME12-5, ME12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution, accepting 85% or 86%	2
• Evaluates correct expression for Z	1

Sample Answer:

For sample of 100, $\mu = 0{\cdot}3$, and $\sigma^2 = \dfrac{0{\cdot}3 \times 0{\cdot}7}{100} = 0{\cdot}0021$. Now, $\hat{P} \approx N(0{\cdot}3, 0{\cdot}0021)$.

Let $z = \dfrac{0{\cdot}25 - 0{\cdot}3}{\sqrt{0{\cdot}0021}} \approx -1{\cdot}09$.

$$\text{Now,}\quad P(\hat{P} > 0{\cdot}25) \approx P(Z > -1{\cdot}09) \approx P(Z < 1{\cdot}09)$$

From Table 1 the probability lies between $P(Z < 1{\cdot}0) \approx 84{\cdot}1\%$ and $P(Z < 1{\cdot}1) \approx 86{\cdot}4\%$ but clearly closer to the latter value.
So there is approximately an 86% chance that 25% of the sampled points are forested.

13(c) (ii) (3 marks)
Outcomes Assessed: *ME12-5, ME12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• Provides correct solution	3
• Approximates $\dfrac{0{\cdot}05n}{\sqrt{0{\cdot}21n}}$ to $1{\cdot}7$ (or similar calculator method)	2
• Finds correct expression for Z	1

Sample Answer:

Let $z = \dfrac{0{\cdot}25n - np}{\sqrt{npq}} = \dfrac{0{\cdot}25n - 0{\cdot}3n}{\sqrt{0{\cdot}21n}} = \dfrac{-0{\cdot}05n}{\sqrt{0{\cdot}21n}}$.

So, we need to solve $P\left(Z > \dfrac{-0{\cdot}05n}{\sqrt{0{\cdot}21n}}\right) = 0{\cdot}955$. That is, $P\left(Z < \dfrac{0{\cdot}05n}{\sqrt{0{\cdot}21n}}\right) = 0{\cdot}955$.

From the z-score table, $P(Z < 1{\cdot}7) \approx 0{\cdot}955$.

Taking $Z \approx 1{\cdot}7$:

$$\frac{0{\cdot}05n}{\sqrt{0{\cdot}21n}} \approx 1{\cdot}7 \qquad \frac{0{\cdot}0025n^2}{0{\cdot}21n} \approx 2{\cdot}89$$

$$n \approx \frac{2{\cdot}89 \times 0{\cdot}21}{0{\cdot}0025} \approx 243$$

So, Carolyn needs to sample 243 points for there to be at least a 95·5% probability that at least 25% of the points are forested.

Question 14 (15 marks)

14(a) (i) (2 marks)
Outcomes Assessed: *ME11-2, ME11-4, ME12-2*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	2
• Finds $t=0$ or $t=4\sin\alpha$	1

Sample Answer:

The projectile lands when vertical position is zero, that is

$$-5t^2+20t\sin\alpha=0$$
$$-5t\,(t-4\sin\alpha)=0$$
$$t=0 \text{ or } 4\sin\alpha$$

Now, the horizontal position x m of the canoe is given by $x=30+3t$ and the horizontal position x m of the projectile is given by $x=20t\cos\alpha$, so the projectile will land in the canoe if these expressions are equal when $t=4\sin\alpha$, so

$$20\,(4\sin\alpha)\cos\alpha=30+3\,(4\sin\alpha)$$
$$20\,(2\sin\alpha\cos\alpha)=15+6\sin\alpha$$

Which gives $20\sin2\alpha-6\sin\alpha-15=0$ as required.

14(a) (ii) (1 mark)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E2*

Criteria	Mark
• Provides correct solution, either by long division or other valid method	1

Sample Answer:

$$\begin{array}{r|l}
 & 5x^2+4x-3 \\ \hline
3x^2+16x-5 & 15x^4+92x^3+30x^2-68x+15 \\
 & \underline{15x^4+80x^3-25x^2} \\
 & \qquad 12x^3+55x^2-68x \\
 & \qquad \underline{12x^3+64x^2-20x} \\
 & \qquad\qquad -9x^2-48x+15 \\
 & \qquad\qquad \underline{-9x^2-48x+15} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Hence, $15x^4+92x^3+30x^2-68x+15=(3x^2+16x-5)(5x^2+4x-3)$.
$\therefore a=5,\ b=4,\ c=-3$.

14(a) (iii) (3 marks)
Outcomes Assessed: *ME12-3*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Eliminates negative values of x and provides correct solutions for α	3
• Uses quadratic formula or otherwise to solve each quadratic factor $= 0$	2
• Correctly uses half-angle substitution for x into an equation in α	1

Sample Answer:

Making the t-substitutions (using x), $x = \tan\left(\frac{\alpha}{2}\right)$ gives $\sin\alpha = \frac{2x}{1+x^2}$ and $\cos\alpha = \frac{1-x^2}{1+x^2}$.

Substituting into the equation $20 \times 2\sin\alpha\cos\alpha - 6\sin\alpha - 15 = 0$ from (i) gives:

$$20 \times 2 \times \frac{2x}{1+x^2} \times \frac{1-x^2}{1+x^2} - 6 \times \frac{2x}{1+x^2} - 15 = 0$$

$$80\left(x-x^3\right) - 12\left(x+x^3\right) - 15\left(1+x^2\right)^2 = 0$$

$$80x - 80x^3 - 12x - 12x^3 - 15 - 30x^2 - 15x^4 = 0$$

$$15x^4 + 92x^3 + 30x^2 - 68x + 15 = 0$$

$$(3x^2+16x-5)(5x^2+4x-3) = 0 \qquad \text{from factorisation in (ii).}$$

Therefore, $x = \dfrac{-16 \pm \sqrt{16^2 - 4\times 3\times(-5)}}{6}$ or $\dfrac{-4 \pm \sqrt{4^2 - 4\times 5\times(-3)}}{10}$

But since α is acute and $x = \tan\left(\frac{\alpha}{2}\right)$, we can eliminate negative values of x:

$$\alpha = 2\tan^{-1}\left(\frac{-8+\sqrt{79}}{3}\right) \text{ or } 2\tan^{-1}\left(\frac{-2+\sqrt{19}}{5}\right) \approx 33^\circ \text{ or } 51^\circ.$$

14(b) (i) (2 marks)
Outcomes Assessed: *ME11-1, ME11-2*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	2
• Finds $P(x)$ and $P(-x)$ in terms of α, β and γ	1

Sample Answer:

$$\begin{aligned}
\text{Let } P(x) &= (x-\alpha)(x-\beta)(x-\gamma) \\
\text{So } P(-x) &= (-x-\alpha)(-x-\beta)(-x-\gamma) \\
&= -(x+\alpha)(x+\beta)(x+\gamma) \\
\text{So } P(x)P(-x) &= -(x-\alpha)(x-\beta)(x-\gamma)(x+\alpha)(x+\beta)(x+\gamma) \\
&= -(x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)
\end{aligned}$$

which is a cubic in x^2 with zeros α^2, β^2 and γ^2.
So $Q(x^2) = -P(x)P(-x)$ is a cubic with zeros α^2, β^2 and γ^2.

14(b) (ii) (2 marks)
Outcomes Assessed: *ME11-1, ME12-1*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• Provides correct solution	2
• Correctly expands $Q(x^2)$	1

Sample Answer:

$$\begin{aligned}Q(x^2) &= -(x^3+kx^2+mx+n)(-x^3+kx^2-mx+n)\\ &= (x^3+kx^2+mx+n)(x^3-kx^2+mx-n)\\ &= x^6-kx^5+mx^4-nx^3+kx^5-k^2x^4+kmx^3-knx^2\\ &\qquad +mx^4-kmx^3+m^2x^2-mnx+nx^3-knx^2+mnx-n^2\\ &= x^6-(k^2-2m)x^4+(m^2-2kn)x^2-n^2\end{aligned}$$

So $Q(x) = x^3-(k^2-2m)x^2+(m^2-2kn)x-n^2$

14(c) (i) (2 marks)
Outcomes Assessed: *ME11-1, ME11-2*
Targeted Performance Bands: *E4*

Criteria	Marks
• Provides correct solution	2
• Evaluates $Q(x)$ correctly	1

Sample Answer:

$P(x) = x^3-3x^2-x+1$ with zeros α, β and γ.
Let $Q(x)$ be a polynomial with zeros α^2, β^2 and γ^2.
Using the process in part (b) with $k=-3$, $m=-1$, and $n=1$ will give:

$$\begin{aligned}Q(x) &= x^3-\left((-3)^2-2\times -1\right)x^2+\left((-1)^2-2\times -3\times 1\right)x-1^2\\ &= x^3-11x^2+7x-1\end{aligned}$$

Applying this process again, let $R(x)$ be a polynomial with zeros α^4, β^4 and γ^4, with the inputs from $Q(x)$ above being $k=-11$, $m=7$, and $n=-1$ will give:

$$\begin{aligned}R(x) &= x^3-\left((-11)^2-2\times 7\right)x^2+\left((7)^2-2\times -11\times -1\right)x-(-1)^2\\ &= x^3-107x^2+27x-1\end{aligned}$$

14(c) (ii) (1 mark)
Outcomes Assessed: *ME11-2, ME11-7*
Targeted Performance Bands: *E3-E4*

Criteria	Mark
• Provides a sufficiently detailed justification capturing most of the solution below	1

Sample Answer:
The sum of zeros of $R(x)$ will give $\alpha^4 - \beta^4 + \gamma^4 = -(-107)/1 = 107$.
Since $|\alpha| > |\beta|$, and $|\alpha| > |\gamma|$, then α^4 is significantly greater than β^4 and γ^4.
So, $\alpha^4 + \beta^4 + \gamma^4 \approx \alpha^4$, so $\alpha^4 \approx 107$.

14(c) (iii) (2 marks)
Outcomes Assessed: *ME11-1, ME11-2*
Targeted Performance Bands: *E4*

Criteria	Marks
• Provides approximation for α AND $P(\alpha)$, and explains what this means	2
• Provides approximation for α OR $P(\alpha)$	1

Sample Answer:
From above, $\alpha \approx \sqrt[4]{107} \approx 3{\cdot}22$.
Also $P\left(\sqrt[4]{107}\right) \approx 0{\cdot}0204$, which means that this is a good approximation for a zero of the polynomial.