Neap

HSC Trial Examination 2020

Mathematics Extension 1

Solutions and marking guidelines

Section I

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 1 **A** If α and $\alpha + 1$ are zeros of $P(x)$, then $P(x) = x^2 - (2\alpha + 1)x + (\alpha^2 + \alpha)$. Equating coefficients gives $b = -(2\alpha + 1)$ and $c = \alpha^2 + \alpha$.	ME-F2 Polynomials ME11–1 Bands E2–E3
Question 2 **D** $\frac{d}{dx}(\tan^{-1} f(x)) = \frac{f'(x)}{1 + (f(x))^2}$ $f(x) = 2x - 1$ and $f'(x) = 2$ $\frac{d}{dx}(\tan^{-1}(2x - 1)) = \frac{2}{1 + (2x - 1)^2}$ $= \frac{2}{4x^2 - 4x + 2}$ $= \frac{1}{2x^2 - 2x + 1}$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 3 **B** Let X represent the number of tails where $X \sim \text{Bin}(3, p)$ and let p represent the probability of obtaining tails. From the frequency distribution, it is clear that $p < 0.5$. Consider: $\{1000P(X = 0), 1000P(X = 1), 1000P(X = 2), 1000P(X = 3)\}$ For $p = 0.3$, the theoretical frequency distribution is $\{343, 441, 189, 27\}$, and for $p = 0.4$ it is $\{216, 432, 288, 64\}$. Compared to the given experimental frequency distribution, the closest theoretical distribution is for $p = 0.4$.	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3
Question 4 **A** $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$ $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$ Equating coefficients of $\sin x$ gives $R\cos\alpha = 1$. (1) Equating coefficients of $\cos x$ gives $R\sin\alpha = 1$. (2) Squaring both (1) and (2) and adding gives $R^2 = 2 \Rightarrow R = \sqrt{2}$ (> 0). Substituting into (1) and (2) gives $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$. So $\alpha = \frac{\pi}{4}$ and hence $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.	ME-T3 Trigonometric Equations ME12–3 Bands E2–E3

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Question 5 **A** The table outlines the possible seating arrangements. M1: 1, M2: 3, M3: 2, M4: 1, F1: 4, F2: 3, F3: 2, F4: 1 Therefore the number of possible seating arrangements is $3! \times 4! = 144$.	ME-A1 Working with Combinatorics ME11–5, ME11–7 Bands E2–E3
Question 6 **C** At $(0, 0)$, $\frac{dy}{dx} = 0$ and so **A** is incorrect. At $(-1, 1)$, $\frac{dy}{dx} = 0$ and so **B** and **D** are incorrect.	ME-C3 Applications of Calculus ME12–4 Bands E2–E3
Question 7 **B** $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$ $= \frac{1}{2}\overrightarrow{OB} + \overrightarrow{AB}$ $= \frac{1}{2}\overrightarrow{OB} + \overrightarrow{AO} + \overrightarrow{OB}$ $= \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b}$ $= \frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3
Question 8 **D** $\int \frac{1}{\sqrt{4-9x^2}}dx = \int \frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)}}dx$ Consider integrals of the form $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$ with $a^2 = \frac{4}{9} \Rightarrow a = \frac{2}{3}$ (> 0). $\frac{1}{3}\sin^{-1}\frac{x}{\frac{2}{3}} = \frac{1}{3}\sin^{-1}\frac{3x}{2}$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 9 **C** The parametric equations are: $x = \cos^2 t$ (1) $y = 4\sin^2 t$ (2) $\frac{(2)}{4}$ gives $\frac{y}{4} = \sin^2 t$. (3) (1) + (3) and using $\cos^2 t + \sin^2 t = 1$ gives $x + \frac{y}{4} = 1 \Rightarrow 4x + y = 4$. $0 \le \cos^2 t \le 1$ and so $0 \le x \le 1$. Therefore, $y = 4 - 4x$ for $0 \le x \le 1$.	ME-F1 Further Work with Functions ME11–2 Bands E2–E3

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 10 **C** The diagonals of $OABC$ are given by $\overrightarrow{OB}$ and $\overrightarrow{CA}$. To prove they are perpendicular, form $\overrightarrow{CA} \cdot \overrightarrow{OB}$ and show that it equals zero. $\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$ and $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$. Therefore $\overrightarrow{CA} \cdot \overrightarrow{OB} = 0$ if $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$.	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3

Section II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) (i) $f(x) = x^2 - 4x + 6$ is a parabola. Excluding the turning point at (2, 2), for each value of $f(x)$ in the range there are two x-values. Geometrically, this corresponds to a horizontal line intersecting the graph twice. If x and y are swapped, each x-value in the domain will have two y-values. Hence the inverse will not be a function.	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • Explains using the horizontal line test OR equivalent merit 1
(ii) Use the completing the square method to express $f(x)$ in turning point form: $f(x) = x^2 - 4x + 6 \qquad (x \le 2)$ $= (x-2)^2 + 2$ Swap x and y, then make y the subject. $x = (y-2)^2 + 2$ $x - 2 = (y-2)^2$ $y - 2 = -\sqrt{x-2}$ ($\sqrt{x-2}$ is discarded as $y \le 2$) $y = -\sqrt{x-2} + 2$ $f^{-1}(x) = -\sqrt{x-2} + 2$	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • Gives the correct solution . . . . .2 • Swaps x and y OR equivalent merit. 1
(iii) The domain is $x \ge 2$ as $x - 2 \ge 0$. The range is $y \le 2$ as $-\sqrt{x-2} \le 0$.	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • States correct domain AND range. . . . . . .2 • States correct domain OR range 1
(iv) The curves $y = f(x)$ and $y = f^{-1}(x)$ have a common intersection with the line $y = x$. For example, attempting to solve $f(x) = x$ for x: $x^2 - 4x + 6 = x$ $x^2 - 5x + 6 = 0$ $x = 2, 3$ When $x = 2$, $y = 2$ and so $(2, 2)$ lies on the line $y = x$. When $x = 3$, $y = 1$ and so $(3, 1)$ does not lie on the line $y = x$. Therefore the coordinates of P are $(2, 2)$.	ME-F1 Further Work with Functions ME11–1 Bands E2–E3 • Gives the correct solution . . . . .2 • Attempts to solve $f(x) = x$ for x OR equivalent merit. .1

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(b) Let $u = 1 + 2\tan x$. $\frac{du}{dx} = 2\sec^2 x = \frac{2}{\cos^2 x} \Rightarrow dx = \frac{\cos^2 x}{2}du$ When $x = 0$, $u = 1$ and when $x = \frac{\pi}{4}$, $u = 3$. $\int_0^{\frac{\pi}{4}} \frac{1}{(1+2\tan x)^2 \cos^2 x}dx = \int_1^3 \frac{1}{2u^2}du$ $= -\left[\frac{1}{2u}\right]_1^3$ $= -\left(\frac{1}{6} - \frac{1}{2}\right)$ $= \frac{1}{3}$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E3 • Gives the correct solution. . . . . 2 • Finds an expression for the integral in terms of u OR equivalent merit 1
(c) Substituting $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\frac{1}{2}x$ into $\cos x - \sin x = 1$ and expressing $1 = \frac{1+t^2}{1+t^2}$ gives: $\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = \frac{1+t^2}{1+t^2}$ $\frac{1-t^2-2t-1-t^2}{1+t^2} = 0$ $\frac{-2(t^2+t)}{1+t^2} = 0$ $t^2 + t = 0$ $t(t+1) = 0$ $t = -1, 0$ $\tan\frac{1}{2}x = -1, 0$ $\tan\frac{1}{2}x = 0 \Rightarrow \frac{1}{2}x = 0, \pi$ $\tan\frac{1}{2}x = -1$ tan is negative in the second quadrant and the related angle is $\frac{\pi}{4}$. $\tan\frac{1}{2}x = -1 \Rightarrow \frac{x}{2} = \frac{3\pi}{4}$ So $x = 0, \frac{3\pi}{2}, 2\pi$.	ME-T3 Trigonometric Equations ME12–3 Bands E2–E3 • Gives the correct solution. . . . . 3 • Determines that $\tan\frac{1}{2}x = -1, 0$ . . . . 2 • Attempts to form a quadratic equation in t 1

 TENME1_SS_20.FM

	Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d) (i)	Substituting $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ into $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$ gives: $W = \underset{\sim}{F} \cdot \underset{\sim}{s}$ $= \begin{pmatrix} 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= 20$	ME-V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution 1
(ii)	A unit vector in the direction of $\overrightarrow{PQ}$ is $\hat{\underset{\sim}{s}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Substituting $\underset{\sim}{F} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, $\hat{\underset{\sim}{s}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\lvert\underset{\sim}{s}\rvert = 5$ into $W = (\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\lvert\underset{\sim}{s}\rvert$ gives: $W = \left(\begin{pmatrix} 4 \\ -2 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right)5$ $= 20$	ME-V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution 1
(iii)	The component of $\underset{\sim}{F}$ in the direction of l is given by $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s}$. Substituting $\underset{\sim}{F} \cdot \underset{\sim}{s} = 20$, $\underset{\sim}{s} \cdot \underset{\sim}{s} = 25$ and $\underset{\sim}{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ into $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s}$ gives: $\left(\frac{\underset{\sim}{F} \cdot \underset{\sim}{s}}{\underset{\sim}{s} \cdot \underset{\sim}{s}}\right)\underset{\sim}{s} = \frac{20}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= \frac{4}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ $= \begin{pmatrix} 2.4 \\ -3.2 \end{pmatrix}$ Alternatively, the component of $\underset{\sim}{F}$ in the direction of l is $(\underset{\sim}{F} \cdot \hat{\underset{\sim}{s}})\hat{\underset{\sim}{s}}$.	ME-V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) (i) Using $\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$ with $A=B=\frac{\pi x}{8}$ gives: $\text{LHS}=\sin\frac{\pi x}{8}\sin\frac{\pi x}{8}$ $=\sin^2\frac{\pi x}{8}$ $\text{RHS}=\frac{1}{2}\left[\cos\left(\frac{\pi x}{8}-\frac{\pi x}{8}\right)-\cos\left(\frac{\pi x}{8}+\frac{\pi x}{8}\right)\right]$ $=\frac{1}{2}\left(\cos 0-\cos\frac{\pi x}{4}\right)$ $=\frac{1}{2}\left(1-\cos\frac{\pi x}{4}\right)$ So $\sin^2\left(\frac{\pi x}{8}\right)=\frac{1}{2}\left(1-\cos\frac{\pi x}{4}\right)$.	ME-T2 Further Trigonometric Identities ME11–3 Bands E2–E3 • Demonstrates that LHS = $\sin^2\frac{\pi x}{8}$. AND • Demonstrates that RHS = $\frac{1}{2}\left(1-\cos\frac{\pi x}{4}\right)$ . . . . 2 • Demonstrates that LHS = $\sin^2\frac{\pi x}{8}$. OR • Demonstrates that RHS = $\frac{1}{2}\left(1-\cos\frac{\pi x}{4}\right)$ 1
(ii) $A=6\int_0^8\sin^2\left(\frac{\pi x}{8}\right)dx$ $=3\int_0^8 1-\cos\frac{\pi x}{4}dx$ $=3\left[x-\frac{4}{\pi}\sin\frac{\pi x}{4}\right]_0^8$ $=3\left(8-\frac{4}{\pi}\sin 2\pi-(0-\sin 0)\right)$ $=3(8-0)$ $=24$	ME-C2 Further Calculus Skills ME12–1, 12–4 Bands E2–E3 • Gives the correct solution. . . . . 2 • Uses the part (a) (i) result to form a definite integral 1
(b) (i) $\text{E}(\hat{P})=p$ $=0.36$ $\text{sd}(\hat{P})=\sqrt{\frac{0.36\times 0.64}{25}}$ $=0.096$	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3 • Correctly shows the mean AND standard deviation. . . . . 2 • Correctly shows the mean OR standard deviation. 1
(ii) Transforming to a standard normal variable, Z, gives: $P\left(Z<\frac{0.12-0.36}{0.096}\right)=P(Z<-2.5)$ $=1-P(Z<2.5)$ $=1-0.9938$ $=0.0062$	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct solution. . . . . 2 • Calculates $z=\frac{0.12-0.36}{0.096}$. OR • Uses the table appropriately with an incorrect value for z. 1

 TENME1_SS_20.FM

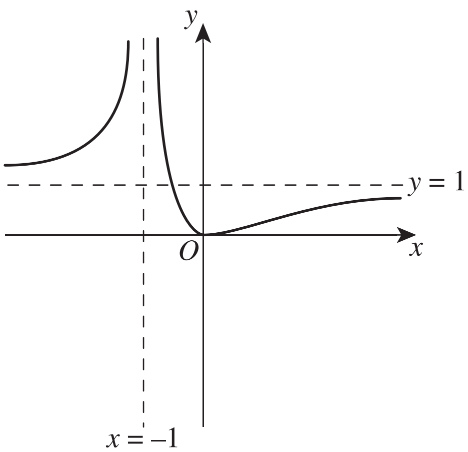
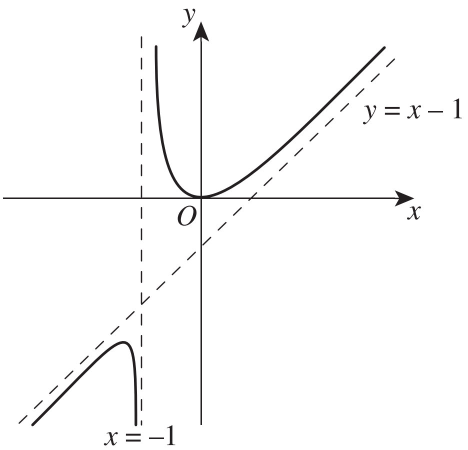
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) The number of adults in the sample who have a mortgage is $25 \times 0.36 = 9$. Let X represent the number of adults who have a mortgage and $X \sim \text{Bin}(25, 0.36)$. $P(X = 9) = \binom{25}{9}(0.36)^9(1 - 0.36)^{16}$ $= 0.1644$	ME-S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct solution . . . . .2 • Attempts to find $P(X = 9)$ where $X \sim \text{Bin}(25, 0.36)$ 1
(c) (i) Rearranging $y = \frac{1}{x^2 + 1}$ to express x^2 in terms of y gives $x^2 = \frac{1}{y} - 1$. $V = \pi\int_{\frac{1}{2}}^{1}\left(\frac{1}{y} - 1\right)dy$ $= \pi\Big[\ln\lvert y\rvert - y\Big]_{\frac{1}{2}}^{1}$ $= \pi\left(\ln 1 - 1 - \left(\ln\frac{1}{2} - \frac{1}{2}\right)\right)$ $= \pi\left(\ln 2 - \frac{1}{2}\right)$	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution . . . . .2 • Provides correct integrand for volume of revolution 1
(ii) Rearranging $y = 1 - \frac{x}{2}$ to express x in terms of y gives $x = 2(1 - y)$. $V = \pi\int_{\frac{1}{2}}^{1}(4(1 - y)^2)dy$ $= -\frac{4\pi}{3}\Big[(1 - y)^3\Big]_{\frac{1}{2}}^{1}$ $= -\frac{4\pi}{3}\left(0 - \frac{1}{8}\right)$ $= \frac{\pi}{6}$ Alternatively: The solid formed is a cone of radius 1 and height $\frac{1}{2}$. Substituting these values into $V = \frac{1}{3}\pi r^2 h$ gives: $V = \frac{1}{3} \times \pi \times 1^2 \times \frac{1}{2}$ $= \frac{\pi}{6}$	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution . . . . .2 • Provides correct integrand for volume of revolution OR equivalent merit. 1
(iii) From the diagram, it can be reasoned that $\pi\left(\ln 2 - \frac{1}{2}\right) > \frac{\pi}{6}$. So $\ln 2 - \frac{1}{2} > \frac{1}{6} \Rightarrow \ln 2 > \frac{2}{3}$.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	
(a) (i) Substituting $x = 12$, $V = 21$ and $t = T$ into $x = Vt\cos\theta$ gives $12 = 21T\cos\theta \Rightarrow T = \dfrac{12}{21\cos\theta}$. Cancelling and using $\dfrac{1}{\cos\theta} = \sec\theta$ gives $T = \dfrac{4}{7}\sec\theta$.	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct solution. 1
(ii) Substituting $y = 2$, $V = 21$ and $t = T$ into $y = Vt\sin\theta - \dfrac{1}{2}gt^2$ gives $2 = 21T\sin\theta - 4.9T^2$. Substituting $T = \dfrac{4}{7}\sec\theta$ into $2 = 21T\sin\theta - 4.9T^2$ gives: $2 = 21\left(\dfrac{4}{7}\sec\theta\right)\sin\theta - 4.9\left(\dfrac{4}{7}\sec\theta\right)^2$ $= 12\tan\theta - \dfrac{8}{5}\sec^2\theta$ $= 12\tan\theta - \dfrac{8}{5}(1 + \tan^2\theta)$ $10 = 60\tan\theta - 8(1 + \tan^2\theta)$ $0 = 8\tan^2\theta - 60\tan\theta + 18$ So $4\tan^2\theta - 30\tan\theta + 9 = 0$.	ME-V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct solution. . . . . 2 • Substitutes $T = \dfrac{4}{7}\sec\theta$ into $2 = 21T\sin\theta - 4.9T^2$ and attempts to form a quadratic in $\tan\theta$ 1
(iii) Using the quadratic formula to solve $4\tan^2\theta - 30\tan\theta + 9 = 0$ for $\tan\theta$ gives $\tan\theta = \dfrac{15 \pm 3\sqrt{21}}{4}\ (= 0.3130\ldots, 7.1869\ldots)$. The shortest flight time occurs for $\theta = \tan^{-1}\left(\dfrac{15 - 3\sqrt{21}}{4}\right)\ (= 0.3130\ldots)$. Substituting $\theta = \tan^{-1}\left(\dfrac{15 - 3\sqrt{21}}{4}\right)\ (= 0.3130\ldots)$ into $T = \dfrac{4}{7}\sec\theta$ gives $T = 0.60$ (s).	ME-V1 Introduction to Vectors ME12–2, 12–6 Bands E2–E3 • Gives the correct solution. . . . . 2 • Correctly solves quadratic equation to obtain two values for $\tan\theta$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Consider $n = 1$. $\text{LHS} = \frac{2}{1 \times 3} = \frac{2}{3}$ and $\text{RHS} = \frac{3}{2} - \frac{2(1)+3}{(1+1)(1+2)} = \frac{4}{6} = \frac{2}{3} = \text{LHS}$. The statement is true when $n = 1$. Suppose true for $n = k$. So $\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} = \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)}$. Show it is true for $n = k + 1$; that is, $\frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)} = \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$ $\text{LHS} = \frac{2}{1 \times 3} + \frac{2}{2 \times 4} + \frac{2}{3 \times 5} + \ldots + \frac{2}{k(k+2)} + \frac{2}{(k+1)(k+3)}$ $= \frac{3}{2} - \frac{2k+3}{(k+1)(k+2)} + \frac{2}{(k+1)(k+3)}$ $= \frac{3}{2} - \frac{(2k+3)(k+3) - 2(k+2)}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2k^2 + 7k + 5}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2k+5}{(k+2)(k+3)}$ $= \frac{3}{2} - \frac{2(k+1)+3}{((k+1)+1)((k+1)+2)}$ $= \text{RHS}$ If true for $n = k$, then true for $n = k + 1$. Hence, by mathematical induction, true for $n \geq 1$.	ME-P1 Proof by Mathematical Induction ME12–1 Bands E2–E4 • Gives the correct proof . . . . .3 • Establishes the inductive step OR equivalent merit. . . . . .2 • Establishes the $n = 1$ case OR equivalent merit. 1

	Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i)	Use of $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ with $A=B=\theta$. $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ Use of $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ with $A=2\theta$ and $B=\theta$. $\tan 3\theta=\dfrac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$ $=\dfrac{\dfrac{2\tan\theta}{1-\tan^2\theta}+\tan\theta}{1-\dfrac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$ $=\dfrac{\dfrac{2\tan\theta+\tan\theta(1-\tan^2\theta)}{1-\tan^2\theta}}{\dfrac{(1-\tan^2\theta)-2\tan^2\theta}{1-\tan^2\theta}}$ $=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$	ME-T3 Trigonometric Equations ME12–3 Bands E2–E4 • Gives the correct proof . . . . 3 • Obtains a correct unsimplified expression for $\tan 3\theta$ involving only $\tan^2\theta$ and $\tan\theta$ OR equivalent merit . . . . 2 • Obtains a correct expression for $\tan 3\theta$ involving only $\tan 2\theta$ and $\tan\theta$ OR equivalent merit. 1
(ii)	Consider $x^3-3x^2-3x+1=0$ with $x=\tan\theta$. $\tan^3\theta-3\tan^2\theta-3\tan\theta+1=0$ $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=1$ So $\tan 3\theta=1$ and finding the roots of $\tan 3\theta=1$ corresponds to finding the roots of the cubic equation where $x=\tan\theta$. $3\theta=\tan^{-1}1+k\pi$ where k is an integer $\theta=\dfrac{\pi}{12}+\dfrac{k\pi}{3}$ $=\dfrac{\pi}{12},\dfrac{5\pi}{12},\dfrac{9\pi}{12}$ $\tan\dfrac{3\pi}{4}=-1$ and so one factor of the cube is $x+1$. So $x^3-3x^2-3x+1=(x+1)(x^2-4x+1)$. So $\tan\dfrac{\pi}{12}$ and $\tan\dfrac{5\pi}{12}$ are the roots of $x^2-4x+1=0$. Solving the quadratic equation $x^2-4x+1=0$ for x gives $x=2\pm\sqrt{3}$. Since $\tan\dfrac{\pi}{12}<\tan\dfrac{5\pi}{12}$, $\tan\dfrac{\pi}{12}$ is the smaller root and $x=2-\sqrt{3}$.	ME-T3 Trigonometric Equations ME12–3 Bands E2–E4 • Gives correct exact value of $\tan\dfrac{\pi}{12}$ . . . . 4 • Gives correct solutions to the cubic equation. . . . . 3 • Deduces that $\theta=\dfrac{\pi}{12}+\dfrac{k\pi}{3}$ where k is an integer OR equivalent merit . . . . . 2 • Deduces that $\tan 3\theta=1$ 1

TENME1_SS_20.FM

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 14	
(a) (i) 	ME-F1 Further Work with Functions ME11–2, 11–7 Bands E2–E4 • Sketches correct graph with asymptotes at $x = -1$ and $y = 1$ . . . . .2 • Shows minimum turning point at origin OR equivalent merit 1
(ii) 	ME-F1 Further Work with Functions ME11–2, 11–7 Bands E2–E4 • Sketches correct graph with asymptotes at $x = -1$ and $y = x - 1$ . . . . .2 • Shows minimum turning point at origin OR equivalent merit 1
(iii) $(f(x))^2 = f(x) \Rightarrow f(x)(f(x) - 1) = 0$ So $f(x) = 1$ or $f(x) = 0$. $-\frac{x}{x+1} = 1 \Rightarrow x = -\frac{1}{2}$ Hence $x = -\frac{1}{2}$ or $x = 0$. **OR** The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect at O, where $x = 0$. The graphs of $y = f(x)$ and $y = (f(x))^2$ intersect on the line $y = 1$, where $x = -\frac{1}{2}$.	ME-F1 Further Work with Functions ME11–2, 11–7 Bands E2–E4 • Gives the correct solution 1
(b) (i) Start with the RHS and show that it equals the LHS. $\text{RHS} = \frac{1}{50}\left(\frac{(50 - A) + A}{A(50 - A)}\right)$ $= \frac{1}{A(50 - A)}$ $= \text{LHS}$	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) This is a differential equation of the form $\frac{dA}{dt} = g(A)$. Attempt to separate variables and integrate both sides. $\int 1\,dt = \int \frac{25}{A(50-A)}dA$ $t = \frac{1}{2}\int\left(\frac{1}{A} + \frac{1}{50-A}\right)dA$ (using the part (i) result) $= \frac{1}{2}(\ln\lvert A\rvert - \ln\lvert 50-A\rvert) + c$ $= \frac{1}{2}\ln\left\lvert\frac{A}{50-A}\right\rvert + c$ Rearranging gives $A_0 e^{2t} = \frac{A}{50-A}$ where $A_0 = e^{-2c}$ and hence $A_0 > 0$. When $t = 0$, $A = \frac{1}{2}$ and so $A_0 = \frac{1}{99}$. *Note: There are various possible ways to find the value of the constant.* $e^{2t} = \frac{99A}{50-A}$ $99Ae^{-2t} = 50 - A$ $A(1 + 99e^{-2t}) = 50$ So $A = \frac{50}{1 + 99e^{-2t}}$.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives correct solution. . . . . 3 • Correctly applies initial condition . . . . . . 2 • Uses the part (b) (i) result and separation of variables to find t in terms of A 1
(iii) As $t \to \infty$, $1 + 99e^{-2t} \to 1$ and so $A \to \frac{50}{1} = 50$. The limiting area of the bacteria colony is 50 cm^2.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) The graph of $\frac{dA}{dt}$ versus A (inverted parabola) has a maximum at $A = 25$. It requires us to find the value of t such that $25 = \frac{50}{1 + 99e^{-2t}}$. $25(1 + 99e^{-2t}) = 50$ $1 + 99e^{-2t} = 2$ $e^{-2t} = \frac{1}{99}$ $e^{2t} = 99$ $t = \frac{1}{2}\ln 99$ (days) The rate of change of the area is at its maximum at $t = \frac{1}{2}\ln 99$ (days). *Note: There are other valid but more time-consuming methods of determining this solution.* *Method 1:* *Finding* $\frac{d^2A}{dt^2} = \frac{1}{25^2}A(50 - A)(50 - 2A)$, *determining that* $\frac{dA}{dt}$ *is a maximum when* $A = 25$ *and then solving for t as above.* *Method 2:* *Determining the value of t when the (non-stationary) point of inflection occurs by finding* $\frac{d^2A}{dt^2}$ *in terms of t and then finding the value of t such that* $\frac{d^2A}{dt^2} = 0$.	ME-C3 Applications of Calculus ME12–4 Bands E2–E4 • Gives the correct solution . . . . .2 • Recognises that the graph of $\frac{dA}{dt}$ versus A has a maximum at $A = 25$ OR equivalent merit. 1
(c) From the table, $f(x) = g^{-1}(x)$ and so $f(-1) = g^{-1}(-1) = 0$. $f'(-1) = \frac{1}{g'(f(-1))}$ $= \frac{1}{g'(0)}$ $= \frac{1}{\frac{1}{2}}$ $= 2$	ME-C2 Further Calculus Skills ME12–1 Bands E2–E4 • Gives the correct solution . . . . .3 • Determines $f(-1) = g^{-1}(-1) = 0$ AND $f'(-1) = \frac{1}{g'(f(-1))}$ . . . . .2 • Determines $f(-1) = g^{-1}(-1) = 0$ OR $f'(-1) = \frac{1}{g'(f(-1))}$ 1