CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2020 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION
MATHEMATICS EXTENSION 2 - MARKING GUIDELINES

Section I
10 marks
Multiple-choice Answer Key

Question	Answer
1	C
2	B
3	D
4	D
5	B
6	A
7	A
8	D
9	C
10	B

Question 1 (1 mark)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E2

Solution	Mark
$\lvert z\rvert=\sqrt{(1)^2+(\sqrt{3})^2}$ $=2$ $Arg(z)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ $=\frac{\pi}{3}$ $\therefore z=2e^{i\frac{\pi}{3}}$ Hence (C)	1

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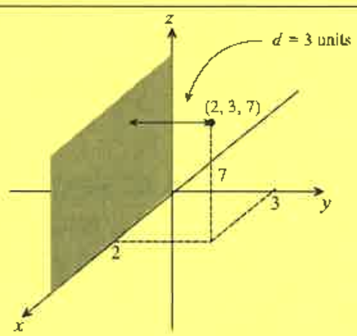
Question 2 (1 mark)
Outcomes Assessed: MEX12-3
Targeted Performance Bands: E2

Solution	Mark
From the diagram, the distance from the $x-z$ plane is the y coordinate. $\therefore d = 3$ units Hence (B)  	1

Question 3 (1 mark)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E2

Solution	Mark
The contrapositive of statement $A \Rightarrow B$ is $\sim B \Rightarrow \sim A$ $\therefore$ the contrapositive of "if n is a prime number, then n is odd" is "if n is not odd, then n is not prime." Hence (D)	1

Question 4 (1 mark)
Outcomes Assessed: MEX12-5
Targeted Performance Bands: E2-E3

Solution	Mark
$\int \frac{1}{x(\log_e x)^2}\,dx = \int \frac{du}{u^2}$ let $u = \log_e x \rightarrow du = \frac{1}{x}dx$ $= \left[-u^{-1}\right]$ $= \frac{-1}{\log_e x} + c$ Hence (D)	1

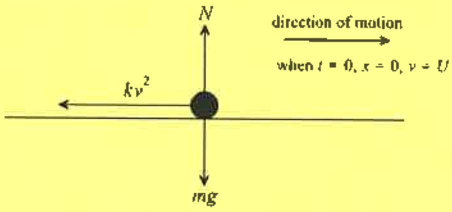
Question 5 (1 mark)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E3

Solution	Mark
$\alpha=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$ in polar form $\alpha^{10}=2^5\left(\cos\left(-\frac{10\pi}{4}\right)+i\sin\left(-\frac{10\pi}{4}\right)\right)$ using De Moivre's Theorem $=32\left(\cos\left(-\frac{5\pi}{2}\right)+i\sin\left(-\frac{5\pi}{2}\right)\right)$ $=-32i$ $\therefore\ \alpha^{10}$ is purely imaginary. Hence (B)	1

Question 6 (1 mark)
Outcomes Assessed: 12-2
Targeted Performance Bands: E3

Solution	Mark
Considering each option: (A) "If a person does not own a pet, then they do not own a cat" does not have a counter-example, as not owning a pet exludes owning any animal. (B) Four points may be joined in a plane in a way that does not form a quadrilateral e.g. joining opposite points to form diagonals. (C) The prime numbers are 2, 3, 5, 7, 11, ... The first prime number is even. (D) Any product of even numbers is even, $\therefore$ if x is even, x^2 is even. $\therefore$ it can be shown simply that if $x=2 \Rightarrow x^2=4$, a counter-example. Hence (A)	1

Question 7 (1 mark)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
The only unbalanced force is the resistance force in the horizontal direction.  $\therefore \ddot{x} = -kv^2$ $\frac{dv}{v^2} = -kdt$ $-\frac{1}{v} = -kt + c$ for $t = 0$, $v = U$ $-\frac{1}{U} = c$ $\therefore -\frac{1}{v} = -kt - \frac{1}{U}$ i.e. $\frac{1}{v} = kt + \frac{1}{U}$ Hence (A)	1

Question 8 (1 mark)

Outcomes Assessed: MEX12-5
Targeted Performance Bands: E3-E4

Solution	Mark
$\int \frac{dx}{\sqrt{8-2x-x^2}} = \int \frac{dx}{\sqrt{9-(x^2+2x+1)}}$ by completing the square $= \int \frac{dx}{\sqrt{9-(x+1)^2}}$ $= \sin^{-1}\left(\frac{x+1}{3}\right) + c$ Hence (D)	1

Question 9 (1 mark)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
The amplitude of motion is $a = 3$ Using the identity $v^2 = n^2\left(a^2 - x^2\right)$ at $x = 0$, $v = \sqrt{3}$ $\left(\sqrt{3}\right)^2 = n^2\left((3)^2 - (0)^2\right)$ $n^2 = \frac{1}{3}$ $\therefore n = \frac{\sqrt{3}}{3}$ $T = \frac{2\pi}{n}$ $= \frac{2\pi}{\frac{\sqrt{3}}{3}}$ $= \frac{6\pi}{\sqrt{3}}$ Hence (C)	1

Question 10 (1 mark)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E4

Solution	Mark
The diagram show the locus of z relative to the complex numbers $0 + i$ and $1 + 0i$. The locus is also the line segment joining the two points. This means that the locus is defined by $\arg\left(\frac{z-i}{z-1}\right) = \pm\pi$ Hence (B)	1

Section II
90 marks

Question 11 (15 marks)

11 (a) (i) (2 marks)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E2

Criteria	Marks
Correct solution	2
Expresses z in modulus-argument form	1

Sample Answer:

$$z = 1 + i$$

$$z = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$\frac{w}{z} = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right]$$

$$= \frac{\sqrt{2}}{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$$

11 (a) (ii) (2 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E2

Criteria	Marks
Correct solution	2
Applies De Moivre's Theorem, or equivalent merit	1

Sample Answer:

$w\bar{z}$ is similar to $\frac{w}{z}$ with a modulus of $\sqrt{2}$:

$$w\bar{z} = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$$

$$\left(w\bar{z}\right)^8 = \left(\sqrt{2}\right)^8\left(\cos\left(8\times\frac{-\pi}{12}\right) + i\sin\left(8\times\frac{-\pi}{12}\right)\right)$$

$$= 16\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -8 - 8\sqrt{3}i$$

11 (b) (2 marks)

Outcomes Assessed: MEX12-3

Targeted Performance Bands: E2

Criteria	Marks
Correct solution	2
Finds the correct magnitudes of the vectors or the dot product	1

Sample Answer:

$$\underset{\sim}{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and } \underset{\sim}{b} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

The angle between two vectors is given by $\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2)(-1) + (1)(1) + (3)(2)$$
$$= 5$$

$$|\underset{\sim}{a}| = \sqrt{(2)^2 + (1)^2 + (3)^2}$$
$$= \sqrt{14}$$

$$|\underset{\sim}{b}| = \sqrt{(-1)^2 + (1)^2 + (2)^2}$$
$$= \sqrt{6}$$

$$\therefore \cos\theta = \frac{5}{\sqrt{14}\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{84}}\right)$$
$$= 56.938...$$
$$= 57° \text{ (nearest degree)}$$

11 (c) (3 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E2-E3

Criteria	Marks
Correct solution	3
Makes some progress towards solution	2
Uses conjugate root theorem	1

Sample Answer:

$P(z) = z^4 - 4z^3 - 3z^2 + 50z - 52$

let $z = 3 - 2i$

If $P(z) = 0$ then $P(\overline{z}) = 0$ (conjugate root theorem)

$z = 3 + 2i$ is also a root of $P(z)$

taking the sum and product of the roots:

$z + \overline{z} = 6$

$$z\overline{z} = |z|^2$$
$$= 13$$

Let the other roots be α and β.

sum of the roots: $\alpha + \beta + 6 = 4 \Rightarrow \alpha + \beta = -2$

product of the roots: $13\alpha\beta = -52 \Rightarrow \beta = -\dfrac{4}{\alpha}$

solving simultaneously:

$$\alpha - \frac{4}{\alpha} = -2$$
$$\alpha^2 - 4 = -2\alpha$$
$$\alpha^2 + 2\alpha - 4 = 0$$
$$\therefore \alpha = -1 \pm \sqrt{5}$$
$$\therefore \beta = -2 - \left(-1 \pm \sqrt{5}\right)$$
$$= -1 \mp \sqrt{5}$$

The solutions of $P(z)$ are $-1+\sqrt{5}, -1-\sqrt{5}, 3-2i, 3+2i$

11 (d) (i) (2 marks)

Outcomes assessed: MEX12-3

Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Finds $\overrightarrow{AB}$ to use in the general form of a straight line in vector form	1

Sample Answer:

$\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{AB} = \underset{\sim}{b} = (2-1)\underset{\sim}{i} + (-1-3)\underset{\sim}{j} + (5--2)\underset{\sim}{k}$

$= \underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$

The equation of the line through AB is given by:

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda_1 \underset{\sim}{b}$ where $\lambda_1 \in \mathbb{R}$

$= \left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) + \lambda_1\left(\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}\right)$

11 (d) (ii) (1 mark)

Outcomes assessed: MEX12-3

Targeted Performance Bands: E3

Criteria	Mark
Correct solution	1

Sample Answer:

The parametric equations of the line are:

$x = 1 + \lambda_1$

$y = 3 - 4\lambda_1$

$z = -2 + 7\lambda_1$

Testing the point $(3, 4, 9)$:

$3 = 1 + \lambda_1$

$\lambda_1 = 2$

substitute $\lambda_1 = 2$ into the other parametric equations:

$y = 3 - 4(2) = -5$

$z = -2 + 7(2) = 12$

$\therefore$ the point $(3, 4, 9)$ does not lie on the line $\underset{\sim}{r} = (\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}) + \lambda_1(\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k})$

11 (d)(iii) (3 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E3

Criteria	Marks
Correct solution	3
Correct substitution without correct geometrical interpretation	2
Progress towards solving the parametric equations	1

Sample Answer:

Equating the parametric equations of the two lines:

$1+\lambda_1 = 1-\lambda_2 \quad (1)$

$3-4\lambda_1 = 2+3\lambda_2 \quad (2)$

$-2+7\lambda_1 = -1+\lambda_2 \quad (3)$

rearranging (1) $\Rightarrow \lambda_1 = -\lambda_2$

substituting into (2):

$3+4\lambda_2 = 2+3\lambda_2$

$\lambda_2 = -1 \quad (\therefore \lambda_1 = 1)$

substitute $\lambda_1 = 1$ and $\lambda_2 = -1$ into (3)

$LHS = -2+7(1) = 5$

$RHS = -1+(-1) = -2$

since $LHS \neq RHS$ the lines do not intersect and are therefore skew.

Question 12 (15 marks)

12 (a) (i) (1 mark)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E2-E3

Criteria	Mark
Correct solution	1

Sample Answer:

completing the square:

$$z^2-2(1+2i)z+(1+2i)^2=-(1+i)+(1+2i)^2$$

$$(z-(1-2i))^2=-1-i+1+4i-4$$

$$\therefore (z-(1-2i))^2=-4+3i$$

12 (a) (ii) (3 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E3

Criteria	Marks
Correct solution	3
Correct solutions of $4x^4+16x^2-9=0$	2
Progress towards finding $\sqrt{-4+3i}$	1

Sample Answer:

Consider $(x+iy)^2=-4+3i$ where $x,y\in\mathbb{R}$

$(x^2-y^2)+2ixy=-4+3i$

equating the real and imaginary parts:

$x^2-y^2=-4 \quad (1)$

$2xy=3 \quad \therefore y=\dfrac{3}{2x} \quad (2)$

substitute (2) into (1)

$$x^2-\frac{9}{4x^2}=-4$$

$$4x^4+16x^2-9=0$$

$$(2x^2+9)(2x^2-1)=0$$

$\therefore x=\pm\dfrac{1}{\sqrt{2}}$ since $x\in\mathbb{R}$ $\quad \therefore y=\dfrac{3\sqrt{2}}{2}$

$\therefore \sqrt{-4+3i}=\dfrac{\sqrt{2}}{2}+\dfrac{3\sqrt{2}}{2}i$ or $-\dfrac{\sqrt{2}}{2}-\dfrac{3\sqrt{2}}{2}i$

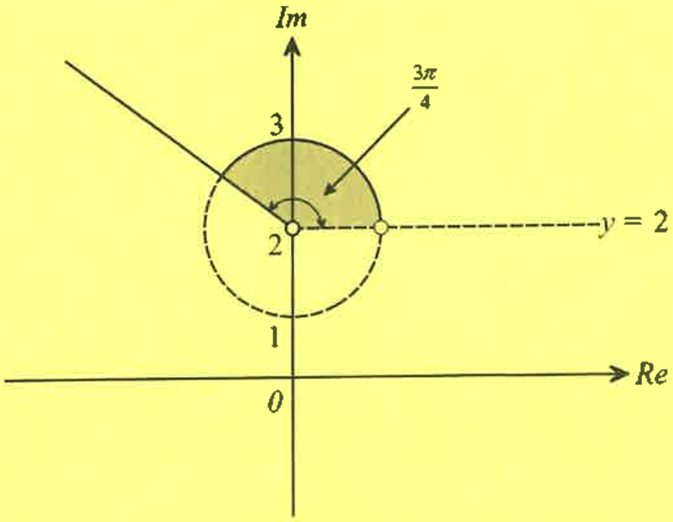
$\therefore z-(1+2i)=\dfrac{\sqrt{2}}{2}+\dfrac{3\sqrt{2}}{2}i$ or $-\dfrac{\sqrt{2}}{2}-\dfrac{3\sqrt{2}}{2}i$

$\therefore z=\left(\dfrac{\sqrt{2}}{2}+1\right)+\left(\dfrac{3\sqrt{2}}{2}+2\right)i$ or $\left(-\dfrac{\sqrt{2}}{2}+1\right)+\left(-\dfrac{3\sqrt{2}}{2}+2\right)i$

12 (b) (2 marks)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
Correct solution	2
Correct sketch of one of the lines and curves	1

Sample Answer:



12(c) (4 marks)

Outcomes Assessed: MEX12-5

Targeted Performance Bands: E2-E3

Criteria	Marks
Correct solution	4
Definite integral correctly integrated	3
Correct new integral	2
Rewrites integrand using substitution	1

Sample Answer:

using $t = \tan\frac{x}{2}$

$8\sin x = 8\left(\frac{2t}{1+t^2}\right)$ and $6\cos x = 6\left(\frac{1-t^2}{1+t^2}\right)$

let $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dx = 2\cos^2\frac{x}{2}\,dt$

$= \frac{2dt}{1+t^2}$

when $x = 0$, $t = \tan\frac{0}{2} = 0$

when $x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{1}{8\sin x + 6\cos x - 10}\,dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\frac{16t}{1+t^2} + \frac{6-6t^2}{1+t^2} - 10} \times \frac{2dt}{1+t^2}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{16t + 6 - 6t^2 - 10\left(1+t^2\right)}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{-8t^2 + 8t - 2}$$

$$= -\frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{4t^2 - 4t + 1}$$

$$= -\frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{(2t-1)^2}$$

$$= -\frac{1}{2}\left[\frac{(2t-1)^{-1}}{-1\times 2}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{4}\left[\frac{1}{2t-1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{1}{4}\left[\frac{1}{\frac{2}{\sqrt{3}} - 1} - (-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2-\sqrt{3}}\right]$$

12(d) (3 marks)
Outcomes Assessed: MEX12-7
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	3
Attempts to solve for two cases of $5-\|x\|$	2
Correct solution in part of the domain	1

Sample Answer:

$$f(|x|)=\frac{4|x|}{5-|x|}$$

to solve $f(|x|)\le 2$ we need to solve the inequation:

$$\frac{4|x|}{5-|x|}\le 2$$

for $5-|x|>0$

$$4|x|\le 2(5-|x|)$$
$$4|x|\le 10-2|x|$$
$$6|x|\le 10$$
$$|x|\le\frac{5}{3}$$
$$\therefore -\frac{5}{3}\le x\le\frac{5}{3}$$

for $5-|x|<0$

$$4|x|\ge 2(5-|x|)$$
$$4|x|\ge 10-2|x|$$
$$6|x|\ge 10$$
$$|x|\ge\frac{5}{3}$$

so $x\le-\frac{5}{3}$ or $x\ge\frac{5}{3}$

but $5-|x|<0$ i.e. $|x|>5$

$\therefore\ x<-5$ or $x>5$

Hence the set of solutions for the inequality $f(|x|)\le 2$ is:

$$x\in(-\infty,-5)\cup\left[-\frac{5}{3},\frac{5}{3}\right]\cup(5,\infty)$$

12 (e) (2 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E2-E3

Criteria	Marks
Correct solution	2
Attempts to prove the statement or substitute values to find a counter-example	1

Sample Answer:

for $\frac{1}{p^2} < \frac{1}{q^2}$ then $p^2 > q^2$ i.e. $p > q$

since this was not stated in the conditions, a counter-example would be

$p = 1, q = 2$

$\frac{1}{p^2} = 1$ and $\frac{1}{q^2} = \frac{1}{4}$

$\therefore \forall p \left(\forall q, \frac{1}{p^2} < \frac{1}{q^2} \right)$ is not true for all p, q real numbers.

Question 13 (15 marks)

13(a) (i) (2 marks)
Outcomes Assessed: MEX12-5
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Correct value of two of the vairables	1

Sample Answer:

$$\frac{-x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{ax+b}{x^2+2}+\frac{c}{1-x}$$

$$-x^2+2x+5 \equiv (ax+b)(1-x)+c(x^2+2)$$

let $x=1$

$$-(1)^2+2(1)+5 \equiv 0+c\left((1)^2+2\right)$$

$$6=3c$$

$$\therefore c=2$$

let $x=0$

$$-(0)^2+2(0)+5 \equiv b+2\left((0)^2+2\right)$$

$$5=b+4$$

$$\therefore b=1$$

let $x=-1$

$$-(-1)^2+2(-1)+5 \equiv (-a+1)(1-(-1))+2\left((-1)^2+2\right)$$

$$2=-2a+2+6$$

$$\therefore a=3$$

$$\therefore \frac{-x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{3x+1}{x^2+2}+\frac{2}{1-x}$$

13 (a) (ii) (2 marks)
Outcomes Assessed: MEX12-5
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
One correct integral	1

Sample Answer:

$$\int \frac{-x^2+2x+5}{\left(x^2+2\right)\left(1-x\right)}\,dx = \int \frac{3x+1}{x^2+2}\,dx + \int \frac{2}{1-x}\,dx$$

$$= \int \frac{3x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx - 2\int \frac{-1}{1-x}\,dx$$

$$= \frac{3}{2}\log_e\left(x^2+2\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - 2\log_e\left(1-x\right) + c$$

13 (b) (i) (1 mark)

Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3

Criteria	Mark
Correct solution	1

Sample Answer:

Since the particle is moving upwards, gravity and air resistance are acting against the direction of motion.

$$\therefore \ddot{x} = -g - 3v$$
$$= -\left(10+3v\right)$$

13 (b) (ii) (3 marks)

Outcomes Assessed: MEX12-6

Targeted Performance Bands: E2-E3

Criteria	Marks
Correct solution	3
Correctly finds integral	2
Correctly uses $\ddot{x} = v\dfrac{dv}{dx}$	1

Sample Answer:

using $\ddot{x} = v\dfrac{dv}{dx}$

$$v\frac{dv}{dx} = -(10+3v)$$

$$\frac{vdv}{10+3v} = -dx$$

$$\frac{1}{3}\left(\frac{10+3v-10}{10+3v}\right)dv = -dx$$

$$\frac{1}{3}\left(1-\frac{10}{10+3v}\right)dv = -dx$$

$$\frac{1}{3}\left(1-\frac{10}{3}\left(\frac{3}{10+3v}\right)\right)dv = -dx$$

integrating both sides:

$$\frac{1}{3}\left(v-\frac{10}{3}\log_e(10+3v)\right) = -x+c$$

for $t=0,\ x=0,\ v=120\text{ ms}^{-1}$

$$\frac{1}{3}\left(120-\frac{10}{3}\log_e(10+3(120))\right) = c$$

$$\therefore c = 40-\frac{10}{9}\log_e 370$$

$$\therefore x = -\frac{1}{3}\left(v-\frac{10}{3}\log_e(10+3v)\right)+40-\frac{10}{9}\log_e 370$$

the maximum height is reached when $v = 0$

$$x_{\max} = -\frac{1}{3}\left(-\frac{10}{3}\log_e(10)\right)+40-\frac{10}{9}\log_e 370$$

$$= \frac{10}{9}\log_e\left(\frac{1}{37}\right)+40$$

$$\approx 36 \text{ metres (nearest metre)}$$

13 (b) (iii) (2 marks)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Uses $\ddot{x} = \dfrac{dv}{dt}$ to find integral relating x and t	1

Sample Answer:

using $\ddot{x} = \dfrac{dv}{dt}$

$$\frac{dv}{dt} = -(10+3v)$$

$$\frac{dv}{10+3v} = -dt$$

integrating both sides:

$$\frac{1}{3}\log_e(10+3v) = -t + c$$

when $t = 0$, $v = 120\ \text{ms}^{-1}$

$$\therefore \frac{1}{3}\log_e(370) = c$$

$$\therefore \frac{1}{3}\log_e(10+3v) = -t + \frac{1}{3}\log_e(370)$$

the maximum height is reached when $v = 0$

$$\frac{1}{3}\log_e(10) = -t + \frac{1}{3}\log_e(370)$$

$$\therefore t = \frac{1}{3}\log_e\left(\frac{370}{10}\right)$$

$$\approx 1.2 \text{ seconds (1 decimal place)}$$

13 (c) (i) (2 marks)
Outcomes Assessed: MEX12-3
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Finds the radius of the sphere	1

Sample Answer:
The radius of the sphere is given by

$$r = |\underset{\sim}{a} - \underset{\sim}{c}|$$
$$= \sqrt{2^2 + 2^2 + 2^2}$$
$$= \sqrt{12}$$

The sphere has equation:

$$(x-2)^2 + (y-2)^2 + (z-2)^2 = 12$$

13 (c) (ii) (3 marks)
Outcomes Assessed: MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
Correct solution	3
Finds correct equation of circle without geometrical description of centre and radius	2
Finds the value of z for which the spheres intersect	1

Sample Answer:

Solving simultaneously:

$$(x-2)^2+(y-2)^2+(z-2)^2=12$$

$$(x-2)^2+(y-2)^2+(z-5)^2=1$$

subtracting equation (2) from (1)

$$(z-2)^2-(z-5)^2=12-1$$

$$(z^2-4z+4)-(z^2-10z+25)=11$$

$$\therefore z=\frac{32}{6}=\frac{16}{3}$$

substitute $z=\dfrac{16}{3}$ into eq (1)

$$(x-2)^2+(y-2)^2+\left(\frac{16}{3}-2\right)^2=12$$

$\therefore (x-2)^2+(y-2)^2=\dfrac{8}{9}$ is the equation of the circle.

This circle has centre $\left(2,2,\dfrac{16}{3}\right)$ and radius $r=\dfrac{\sqrt{8}}{3}$.

Question 14 (15 marks)

14 (a) (3 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
Correct proof	3
Deduces that p is divisible by 5	2
Establishes assumption to prove by contradiction	1

Sample Answer:

Assume $\frac{1+\sqrt{5}}{2}$ is rational, i.e. $\sqrt{5}$ is rational (since $1, 2 \in \mathbb{Q}$)

let $\sqrt{5} = \frac{p}{q}$, where p and q are integers with no common factor, $q \neq 0$.

$$\left(\sqrt{5}\right)^2 = \left(\frac{p}{q}\right)^2$$

$$5 = \frac{p^2}{q^2}$$

$$5q^2 = p^2$$

$\therefore p$ is divisible by 5 i.e. $p = 5k,\ k \in \mathbb{Z}$

$$\therefore 5q^2 = (5k)^2$$

$$q^2 = 5k^2$$

$\therefore q$ is also divisible by 5, a contradiction since p and q have no common factor

$\therefore \sqrt{5}$ is irrational

$\therefore \frac{1+\sqrt{5}}{2}$ is irrational.

14 (b) (i) (3 marks)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	3
Correct amplitude or period	2
Correct transformation of trigonometric function	1

Sample Answer:

Express $\sin\frac{\pi}{7}t+\cos\frac{\pi}{7}t$ in the form $r\sin\left(\frac{\pi}{7}t+\alpha\right)$

$$\sin\frac{\pi}{7}t+\cos\frac{\pi}{7}t=r\left(\sin\left(\frac{\pi}{7}t\right)\sin\alpha+\cos\left(\frac{\pi}{7}t\right)\cos\alpha\right)$$

equating the coefficients of $\sin\frac{\pi}{7}t$ and $\cos\frac{\pi}{7}t$

$r\sin\alpha=1$

$r\cos\alpha=1$

$\therefore r=\sqrt{2}$ and $\alpha=\frac{\pi}{4}$

$$\therefore\ p=1.5+\frac{1}{2}\sin\frac{\pi}{7}t+\frac{1}{2}\cos\frac{\pi}{7}t$$
$$=1.5+\frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{7}t+\frac{\pi}{4}\right)$$

amplitude: $a=\frac{\sqrt{2}}{2}$

period: $T=\frac{2\pi}{\frac{\pi}{7}}=14$ days

14 (b) (ii) (1 mark)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3

Criteria	Mark
Correct solution	1

Sample Answer:
12pm on Monday is 12 hours after midnight Sunday.

$\therefore t = 0.5$ days

$$p = 1.5 + \frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{7}\times 0.5 + \frac{\pi}{4}\right)$$

$= 2.0987...$

$= \$2.10$ (nearest cent)

14 (b) (iii) (3 marks)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3

Criteria	Marks
Correct solution	3
Correct values of t	2
Correct differentiation	1

Sample Answer:

$$\frac{dp}{dt}=\frac{\sqrt{2}}{2}\times\frac{\pi}{7}\cos\left(\frac{\pi}{7}t+\frac{\pi}{4}\right)$$

let $\frac{dp}{dt}=0$ to find stationary points

$$\frac{\sqrt{2}}{2}\times\frac{\pi}{7}\cos\left(\frac{\pi}{7}t+\frac{\pi}{4}\right)=0$$

$$\therefore\frac{\pi}{7}t+\frac{\pi}{4}=\frac{\pi}{2},\frac{3\pi}{2},\ldots$$

$$\therefore t=\frac{7}{4},\frac{35}{4},\ldots$$

$$\frac{d^2p}{dt^2}=-\frac{\sqrt{2}}{2}\times\left(\frac{\pi}{7}\right)^2\sin\left(\frac{\pi}{7}t+\frac{\pi}{4}\right)$$

for $t=\frac{7}{4}$, $\frac{d^2p}{dt^2}=-\frac{\sqrt{2}}{2}\times\left(\frac{\pi}{7}\right)^2$ i.e. price is max when $t=\frac{7}{4}$

for $t=\frac{35}{4}$, $\frac{d^2p}{dt^2}=\frac{\sqrt{2}}{2}\times\left(\frac{\pi}{7}\right)^2$ i.e. price is min when $t=\frac{35}{4}$

$\therefore$ the fuel price is first at a minimum at $t=8.75$ days

i.e. $t=8$ days, 18 hours after midnight Sunday

$\therefore$ fuel price is first at a minimum at 6:00pm on Tuesday week.

14 (c) (i) (2 marks)

Outcomes Assessed: MEX12-6

Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Correctly uses $\ddot{x} = v\dfrac{dv}{dx}$	1

Sample Answer:

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 2x^3 + 6x^2 + 4x$$

$$\int d\left(\frac{1}{2}v^2\right) = \int\left(2x^3 + 6x^2 + 4x\right)dx$$

$$\frac{1}{2}v^2 = \frac{x^4}{2} + 2x^3 + 2x^2 + c$$

$$v^2 = x^4 + 4x^3 + 4x^2 + c$$

when $t = 0,\ x = 1,\ v = -3$

$$(-3)^2 = (1)^4 + 4(1)^3 + 4(1)^2 + c$$

$$9 = 9 + c$$

$$\therefore c = 0$$

$$\therefore v^2 = x^4 + 4x^3 + 4x^2$$

$$= x^2\left(x^2 + 4x + 4\right)$$

$$= x^2\left(x + 2\right)^2$$

$$v = \pm\sqrt{x^2\left(x+2\right)^2}$$

$v = -x\left(x + 2\right)$ since initially $v = -3$

14 (c) (ii) (2 marks)

Outcomes Assessed: MEX12-6

Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Correctly integrates to relate x and t	1

Sample Answer:

$$\frac{dx}{dt}=-x(x+2)$$

$$\frac{dx}{x(x+2)}=-dt$$

using partial fraction decomposition:

$$\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)dx=-dt$$

integrating both sides:

$$\frac{1}{2}\left(\ln x-\ln(x+2)\right)=-t+c$$

$$\frac{1}{2}\ln\left(\frac{x}{x+2}\right)=-t+c$$

when $t=0$, $x=1$

$$\frac{1}{2}\ln\left(\frac{1}{1+2}\right)=0+c$$

$$\therefore c=-\frac{1}{2}\ln 3$$

$$\frac{1}{2}\ln\left(\frac{x}{x+2}\right)=-t-\frac{1}{2}\ln 3$$

$$\frac{1}{2}\ln\left(\frac{x}{x+2}\right)+\frac{1}{2}\ln 3=-t$$

$$\frac{1}{2}\ln\left(\frac{3x}{x+2}\right)=-t$$

$$\ln\left(\frac{x+2}{3x}\right)=2t$$

$$\ln\left(\frac{x}{3x}+\frac{2}{3x}\right)=2t$$

$$\frac{1}{3}+\frac{2}{3x}=e^{2t}$$

$$\frac{2}{3x}=\frac{3e^{2t}-1}{3}$$

$$\therefore x=\frac{2}{3e^{2t}-1}$$

14 (c) (iii) (1 mark)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3

Criteria	Mark
Correct solution	1

Sample Answer:

for $t \to \infty$

$$x \to \lim_{t\to\infty} \frac{2}{3e^{2t}-1}$$

$$\to \frac{2}{\infty}$$

$$\to 0$$

$\therefore$ the limiting position of the particle is $x = 0$.

Question 15 (15 marks)

15 (a) (3 marks)
Outcomes Assessed: MEX12-5
Targeted Performance Bands: E3

Criteria	Marks
Correct solution, or equivalent merit	3
Correctly uses integration by parts and applies it again	2
Attempts to use integration by parts, or equivalent merit	1

Sample Answer:

$I=\int e^{-x}\sin(-x)\,dx$

using by parts integration:

$u=e^{-x} \qquad dv=\sin(-x)\,dx$

$du=-e^{-x}dx \qquad v=\cos(-x)$

$\therefore I=e^{-x}\cos(-x)-\int -e^{-x}\cos(-x)\,dx$

$=e^{-x}\cos(-x)+\int e^{-x}\cos(-x)\,dx$

using a second application of by parts integration:

$I=e^{-x}\cos(-x)+\int e^{-x}\cos(-x)\,dx$

$=e^{-x}\cos(-x)-e^{-x}\sin(-x)-\int(-e^{-x})\times(-\sin(-x))\,dx$

$=e^{-x}\cos x+e^{-x}\sin x-\underbrace{\int e^{-x}\sin(-x)\,dx}_{I}$

$2I=e^{-x}\cos x+e^{-x}\sin x$

$I=\dfrac{e^{-x}\cos x+e^{-x}\sin x}{2}$

$\therefore \int e^{-x}\sin(-x)\,dx=\dfrac{1}{2}e^{-x}(\cos x+\sin x)+c$

15 (b) (i) (2 marks)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	2
Correct use of vertical component	1

Sample Answer:

$$y=\frac{10+0.4\times10\sqrt{3}\times\sin60^\circ}{(0.4)^2}\left(1-e^{-0.4t}\right)-\frac{10t}{0.4}$$

$$=\frac{10+4\sqrt{3}\times\frac{\sqrt{3}}{2}}{\frac{4}{25}}\left(1-e^{-0.4t}\right)-25t$$

$$=\frac{25}{4}(10+6)\left(1-e^{-0.4t}\right)-25t$$

$$=100-100e^{-0.4t}-25t$$

$$\dot{y}=40e^{-0.4t}-25$$

the greatest height is reached when $\dot{y}=0$:

$$0=40e^{-0.4t}-25$$

$$e^{-0.4t}=\frac{25}{40}$$

$$-0.4t=\log_e\left(\frac{5}{8}\right)$$

$$t=-\frac{5}{2}\log_e\left(\frac{5}{8}\right)$$

$$=1.175...$$

$=1.18$ seconds (2 decimal places)

15 (b) (ii) (3 marks)
Outcomes Assessed: MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	3
Correctly finds the magnitude or direction of the velocity	2
Correct value of horizontal component of velocity	1

Sample Answer:

for $t = 2.6$

$$\dot{y} = 40e^{-0.4\times 2.6} - 25$$
$$= -10.86 \text{ ms}^{-1} \ (2 \text{ decimal places})$$

$$\text{also } x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$$
$$= \frac{10\sqrt{3}\cos 60^\circ}{0.4}\left(1 - e^{-0.4t}\right)$$
$$= \frac{25\sqrt{3}}{2}\left(1 - e^{-0.4t}\right)$$
$$\dot{x} = 5\sqrt{3}e^{-0.4t}$$

for $t = 2.6$

$$\dot{x} = 5\sqrt{3}e^{-0.4\times 2.6}$$
$$= 3.06 \text{ ms}^{-1} \text{ (2 decimal places)}$$

$\therefore$ the magnitude of the velocity of the particle is:

$$|\underset{\sim}{v}| = \sqrt{(10.86)^2 + (3.06)^2}$$
$$= 11.28 \text{ ms}^{-1}$$

the direction of the particle is $\tan^{-1}\left(\frac{10.86}{3.06}\right) = 74^\circ$ below the horizontal (to the nearest degree).

15 (c) (i) (2 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E3

Criteria	Marks
Correct solution	2
Correct method in finding roots of unity	1

Sample Answer:

let $z = r(\cos\theta + i\sin\theta)$

$z^5 = 1$ can be written in mod-arg form:

$\left(r(\cos\theta + i\sin\theta)\right)^5 = \cos 0 + i\sin 0$

$r^5(\cos 5\theta + i\sin 5\theta) = \cos 0 + i\sin 0$ using De Moivre's Theorem

$\therefore r = 1$ and $5\theta = 2k\pi \Rightarrow \theta = \dfrac{2k\pi}{5}$

for $k = 0,\ \theta = 0$

for $k = 1,\ \theta = \dfrac{2\pi}{5}$

for $k = 2,\ \theta = \dfrac{4\pi}{5}$

for $k = -1,\ \theta = -\dfrac{2\pi}{5}$

for $k = -2,\ \theta = -\dfrac{4\pi}{5}$

$\therefore$ the roots of $z^5 = 1$ are:

$z_1 = \cos\left(\dfrac{2\pi}{5}\right) + i\sin\left(\dfrac{2\pi}{5}\right)$

$z_2 = \cos\left(\dfrac{4\pi}{5}\right) + i\sin\left(\dfrac{4\pi}{5}\right)$

$z_3 = \cos\left(-\dfrac{2\pi}{5}\right) + i\sin\left(-\dfrac{2\pi}{5}\right)$

$z_4 = \cos\left(-\dfrac{4\pi}{5}\right) + i\sin\left(-\dfrac{4\pi}{5}\right)$

$z_5 = 1$

15 (c) (ii) (2 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	2
Correct use of the conjugates to simplify complex factors	1

Sample Answer:

From part (i) it can be seen that:

$z_4 = \overline{z}_1$ and $z_3 = \overline{z}_2$

using the fundamental theorem of algebra:

$$z^5 - 1 = (z-1)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$$
$$= (z-1)(z-z_1)(z-z_2)(z-\overline{z}_2)(z-\overline{z}_1)$$

using the property $(x-\alpha)(x-\overline{\alpha}) = x^2 - 2\operatorname{Re}(\alpha) + |\alpha|^2$

$$z^5 - 1 = (z-1)\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)+1\right)$$

$$\therefore \frac{z^5-1}{z-1} = \left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)+1\right)$$

15 (c) (iii) (3 marks)

Outcomes Assessed: MEX12-4

Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	3
Progress towards solution	2
Attempts to establish $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$ as roots of a quadratic equation using part (ii)	1

Sample Answer:

The sum of the roots is 0.

$$\therefore z_1 + z_2 + z_3 + z_4 + z_5 = 0$$

$$z_1 + z_2 + \overline{z}_2 + \overline{z}_1 + 1 = 0$$

$$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1$$

$$\therefore \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2} \qquad (1)$$

also, from part (ii):

$$\frac{z^5-1}{z-1} = \left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)+1\right) \quad \text{as } \frac{z^5-1}{z-1} \text{ is the sum of a GP with } a=1, r=z, n=5$$

$$z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)+1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)+1\right)$$

considering the z^2 terms on both sides :

$$LHS = z^2$$

$$RHS = z^2 + 4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) + z^2 = z^2\left(2 + 4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)\right)$$

equating the coefficients:

$$2 + 4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = 1$$

$$\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -\frac{1}{4} \qquad (2)$$

$\therefore$ from (1) and (2), $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$ are the roots of a quadratic of the form:

$$x^2 + \frac{1}{2}x - \frac{1}{4} = 0$$

$$4x^2 + 2x - 1 = 0$$

Question 16 (15 marks)

16 (a) (3 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	3
Significant progress toward completing the inductive step	2
Shows statement is true for $n=3$ and assumes true for $n=k$	1

Sample Answer:

Show the statement is true for $n=1, 2, 3$:

$u_1=(1-1)5^1=0$

$u_2=(2-1)5^2=25$

$$u_3=10u_2-25u_1 = 10\times25-25\times0 = 250 \qquad u_3=(3-1)\times5^3 = 250$$

hence the statement is true for $n=1,2,3$.

Assume the statements are true for $n=k$, that is:

$u_k=(k-1)\times5^k$ and $u_k=10u_{k-1}-25u_{k-2}$

Prove that the statement is true for $n=k+1$, that is, required to prove $u_{k+1}=k\times5^{k+1}$

$$\begin{aligned}u_{k+1}&=10u_k-25u_{k-1}\\&=10(k-1)\times5^k-25(k-2)\times5^{k-1}\\&=2(k-1)\times5^{k+1}-(k-2)\times5^{k+1}\\&=(2k-2-k+2)\times5^{k+1}\\&=k\times5^{k+1}\end{aligned}$$

hence the statement is true for $n=k+1$.

since the statement is true for $n=1$, $n=k$ and $n=k+1$ it is true by mathematical induction for all positive integers $n\geq1$.

16 (b) (i) (4 marks)

Outcomes Assessed: MEX12-5

Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	4
Correctly recognises the equivalent integral on the RHS	3
Correctly uses one application of integration by parts	2
Attempts to use integration by parts	1

Sample Answer:

Let $I_n = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x \sin^2 x \, dx$

let $u = \cos^{n-1} x$ $\qquad$ $dv = \cos x \sin^2 x \, dx$

$du = (n-1)\cos^{n-2} x(-\sin x)$ $\qquad$ $v = \frac{1}{3}\sin^3 x$

using by parts integration:

$$I_n = uv - \int v \, du$$

$$= \left[\frac{1}{3}\sin^3 x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{3}\sin^3 x(n-1)\cos^{n-2} x(-\sin x)\, dx$$

$$= 0 + \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{n-2} x \sin x \, dx$$

$$= \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x \sin^2 x \, dx$$

$$= \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x\left(1-\cos^2 x\right) dx$$

$$= \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x \, dx - \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n} x \, dx$$

$$= \frac{n-1}{3} I_{n-2} - \frac{n-1}{3} I_n$$

$$3I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$3I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n+2} I_{n-2}$$

16 (b) (ii) (1 mark)
Outcomes Assessed: MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Mark
Correctly applies recurrence relation	1

Sample Answer:

$$I_2 = \frac{2-1}{2+2} I_0$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos^0 x \sin^2 x \, dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1-\cos 2x\right) dx$$

$$= \frac{1}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin \pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{16}$$

16 (c) (i) (1 mark)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Mark
Correct solution	1

Sample Answer:
For $x > 1$:

$$x^3 > x^2 > x > 0$$

so $x\sqrt{x} > x > \sqrt{x} > 0$

by taking the reciprocals:

$$0 < \frac{1}{x\sqrt{x}} < \frac{1}{x} < \frac{1}{\sqrt{x}}$$

16 (c)(ii) (1 mark)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Mark
Correct solution	1

Sample Answer:
For $0 < x < 1$:

$$x > x^2 > x^3 > 0$$

so $\sqrt{x} > x > x\sqrt{x} > 0$

by taking reciprocals:

$$0 < \frac{1}{\sqrt{x}} < \frac{1}{x} < \frac{1}{x\sqrt{x}}$$

16 (c)(iii) (2 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	2
Correctly integrates the terms of the inequation	1

Sample Answer:

For $x \geq 1$, $\dfrac{1}{x\sqrt{x}} \leq \dfrac{1}{x} \leq \dfrac{1}{\sqrt{x}}$ with equality when $x = 1$.

Integrating from $x = 1$ to $x = t$

$$\int_1^t \frac{1}{x\sqrt{x}}\,dx \leq \int_1^t \frac{1}{x}\,dx \leq \int_1^t \frac{1}{\sqrt{x}}\,dx$$

$$\left[-2x^{-\frac{1}{2}}\right]_1^t \leq \left[\ln x\right]_1^t \leq \left[2x^{\frac{1}{2}}\right]_1^t$$

$$-2\left[\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{1}}\right] \leq \ln t - \ln 1 \leq 2\left[\sqrt{t} - 1\right]$$

$$\therefore \frac{2\left(\sqrt{t}-1\right)}{\sqrt{t}} \leq \ln t \leq 2\left(\sqrt{t}-1\right).$$

16 (c)(iv) (3 marks)

Outcomes Assessed: MEX12-2

Targeted Performance Bands: E3-E4

Criteria	Marks
Correct solution	3
One limit correctly evaluated	2
Attempts to use the squeeze law of limits or equivalent	1

Sample Answer:

from part (ii) $\dfrac{2\left(\sqrt{t}-1\right)}{\sqrt{t}} \le \ln t \le 2\left(\sqrt{t}-1\right).$

multiplying through by t gives:

$$2\sqrt{t}\left(\sqrt{t}-1\right) \le t\ln t \le 2t\left(\sqrt{t}-1\right)$$

as $t \to 0$

$$2\sqrt{t}\left(\sqrt{t}-1\right) \to 0$$

and $2t\left(\sqrt{t}-1\right) \to 0$

$$\therefore 0 \le \lim_{t\to 0}\left(t\ln t\right) \le 0$$

$$\therefore \lim_{t\to 0}\left(t\ln t\right) = 0$$

from part (ii) $\dfrac{2\left(\sqrt{t}-1\right)}{\sqrt{t}} \le \ln t \le 2\left(\sqrt{t}-1\right).$

dividing through by t gives:

$$\frac{2\left(\sqrt{t}-1\right)}{t\sqrt{t}} \le \frac{\ln t}{t} \le \frac{2\left(\sqrt{t}-1\right)}{t}$$

$$\frac{2}{t} - \frac{2}{t\sqrt{t}} \le \frac{\ln t}{t} \le \frac{2}{\sqrt{t}} - \frac{2}{t}$$

as $t \to \infty$

$$\frac{2}{t} - \frac{2}{t\sqrt{t}} \to 0$$

and $\dfrac{2}{\sqrt{t}} - \dfrac{2}{t} \to 0$

$$\therefore 0 \le \lim_{t\to\infty}\left(\frac{\ln t}{t}\right) \le 0$$

$$\therefore \lim_{t\to\infty}\left(\frac{\ln t}{t}\right) = 0 \quad \text{and} \quad \lim_{t\to 0}\left(t\ln t\right) = 0$$

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